Global Stability for Charged Scalar Fields in Spacetimes close to Minkowski

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Abstract

In this work I prove global stability for the zero solution of the massless Charge-Scalar Field system on a background spacetime which is close to 1+3-dimensional Minkowski space. In particular, my analysis takes place in a class of background metrics which satisfy certain energy and decay estimates consistent with the behavior of small-data solutions to Einstein's Vacuum Equations in harmonic coordinates. My results are analogous to results obtained for the same system in Minkowski space by Lindblad and Sterbenz in [16].

The proof relies on a single-parameter modification of the standard null frame and Lorentz fields which depends on the mass associated with the metric. In particular, the Lorentz fields are the same as those used in [13] and the modified null frame can be written in terms of the modified Lorentz fields in a manner analogous to the standard null frame. This dissertation is based on my paper [9], *Global Stability for Charged Scalar Fields in an Asymptotically Flat Metric in Harmonic Gauge*.

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Contents

1 Introduction

In this paper we prove stability and decay rates for solutions of the massless Einstein-Charge Scalar Field system, also called the massless Einstein-Maxwell-Klein-Gordon system, on an asymptotically flat metric close to Minkowski space.

First, given a background spacetime $(\mathcal{M}, g)$ and a real one-form A, one can define the complex covariant derivative

$$D_\alpha = \nabla_\alpha + iA_\alpha, \tag{1.1}$$

where ∇ is the Levi-Civita connection on g. Then, for the two-form $F = dA$ and a complex scalar function ϕ, the massless Charge-Scalar Field system is defined as follows:

$$D^\alpha D_\alpha \phi = 0, \tag{1.2a}$$

$$\nabla^\beta F_{\alpha\beta} = \Im\left(\phi \overline{D_\alpha \phi}\right), \tag{1.2b}$$

$$\nabla^\beta ({}^*F)_{\alpha\beta} = 0. \tag{1.2c}$$

Here and in what follows, $\Re, \Im$ denote the real and imaginary parts of a quantity respectively.

Given this system along with suitable initial conditions for F and ϕ, one has some freedom in the choice of the potential A which it is not necessary to resolve. These quantities are tied together by the commutator relation

$$[D_\alpha, D_\beta]\phi = iF_{\alpha\beta}\phi. \tag{1.3}$$

The right hand side of (1.2b) is the current vector, J_α, and was selected as such in order to make the combined energy-momentum tensor,

$$Q_{\alpha\beta}[\phi, F] = \Re\left(D_\alpha\phi\overline{D_\beta\phi} - \frac{1}{2}g_{\alpha\beta}D_\gamma\phi\overline{D^\gamma\phi}\right) + F_{\alpha\gamma}F^\gamma_\beta - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}, \tag{1.4}$$

divergence free. We separate this tensor into its scalar and field quantities, respectively

$$Q_{\alpha\beta}[\phi] = \Re\left(D_\alpha\phi\overline{D_\beta\phi} - \frac{1}{2}g_{\alpha\beta}D_\gamma\phi\overline{D^\gamma\phi}\right), \tag{1.5a}$$

$$Q_{\alpha\beta}[F] = F_{\alpha\gamma}F^\gamma_\beta - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}. \tag{1.5b}$$

These satisfy the identities

$$\nabla^\beta Q_{\alpha\beta}[\phi] = F_{\alpha\gamma}J^\gamma, \qquad \nabla^\beta Q_{\alpha\beta}[F] = -F_{\alpha\gamma}J^\gamma, \tag{1.6}$$

which follow from the commutator identity (1.3) along with the identity

$$\nabla_\alpha(\phi\overline{\psi}) = D_\alpha\phi\overline{\psi} + \phi\overline{D_\alpha\psi} \tag{1.7}$$

in the scalar term, and antisymmetry along with the identity

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \tag{1.8}$$

in the electromagnetic term.

We consider spacetimes $(\mathcal{M}, g)$ close to Minkowski, in the sense that they satisfy certain L^2 and L^∞ estimates consistent with small-data solutions to Einstein's Vacuum Equations in harmonic gauge. We in particular assume certain energy norms consistent with the stability result [15], combined with nicer L^∞ estimates on certain components of the metric shown in [13].

There is a natural way to frame these L^∞ results, which comes from the mass corresponding to the metric. We assume the metric is of the form

$$g_{\alpha\beta} = m_{\alpha\beta} + \left(\frac{M\chi}{1+r}\right)\delta_{\alpha\beta} + h_{\alpha\beta}, \tag{1.9}$$

where M is a small constant corresponding to the ADM mass, h is a small $(0,2)$-tensor, and χ is a smooth cutoff function equal to 1 for $r > 3t/4$ and 0 for $r < t/2$, such that $\partial\chi$ decays like t^{-1}.

This metric corresponds to small-data solutions of the Einstein Vacuum Equations, as well as a range of coupled Einstein-field systems. In particular, it is our hope that the stability results here can be used as a priori estimates which can prove the stability of the Einstein-massless Charge Scalar Field system, which in harmonic coordinates can be written as the system (1.2) in a metric satisfying the system

$$\Box_g(g_{\mu\nu}) = F_{\mu\nu}(g)(\partial g, \partial g) + C(Q_{\mu\nu} - \frac{1}{2}\mathrm{tr}(Q)), \tag{1.10}$$

where Q is the quantity in equation (1.4), and F is quadratic in derivatives of g with nice asymptotic properties in the null decomposition of g. This set of nice asymptotic properties is in particular known as the weak null condition.

We can think of small-data solutions of the MKG equations as solutions for the Einstein-Field Equations with small C, so that for at least a long time the system takes place on a metric which is mostly unaffected by solutions of the Charge-Scalar Field system. Global existence will ideally follow from estimates on Q which follow from these results, from the fact that we have nicer energy and decay estimates for this tensor than for terms corresponding with terms on the right hand side coming from g.

We note that in the wave zone $t \approx r$ and in the exterior, our metric behaves similarly to Schwarzschild, which requires some additional geometric consideration even for large r. Here, we draw geometric inspiration from analysis of solutions to wave equations in Schwarzschild carried out by Blue and Sterbenz in [3], where they take the conformal Morawetz estimate with r, ∂_r replaced by the tortoise coordinate and derivative r^* and ∂_{r^*}. We can consider the approximate optical functions $u^* = t + r^*$, $\underline{u}^* = t - r^*$.

In our case, we cannot hope to recover the full conformal Morawetz estimate using only the geometric structure of these approximate optical functions, due to insufficient decay in perturbations of the metric. However, we can establish a fractional Morawetz estimate, analogous to that in [16], for certain fractional weights $u^{*2s}, \underline{u}^{*2s}$ which depend on the initial spatial decay of the metric. We show the base estimate, along with a discussion on why the fractional estimate and curved fields are necessary, in section 2.6.

Our primary tool here is a null decomposition and a set of vector fields which are defined only in terms of the parameter M which behave nicely with respect to the metric $\tilde{m} = m + M\chi/(1+r)\delta$ This set of Lorentz fields was used in [13] to establish nicer asymptotic behavior for certain components of the metric. The null decomposition we use is a natural extension of these fields, and allows us to establish nicer estimates on certain derivatives of ϕ and components of F than the corresponding null fields in Minkowski would.

In order to define these fields, we define the adapted tortoise coordinate,

$$r^* = r + M\chi \ln(1+r),$$

such that $\chi = \tilde{\chi}(r/t)\tilde{\chi}(r)$, where $\tilde{\chi}$ is a smooth increasing function satisfying

$$\tilde{\chi}(y) = \begin{cases} 1 & y \geq 3/4 \\ 0 & y \leq 1/2 \end{cases}.$$

We easily see that, for $t > 1$, $r^* = r$ in the far interior $r < t/2$ and $r^* = r + M\ln(1+r)$ in the extended exterior region, $r > 3t/4$. We can use this quantity to define the modified coordinates

$$u^* = t - r^*, \qquad \underline{u}^* = t + r^*, \qquad t^* = t, \qquad x^{*i} = \omega^i r^*. \tag{1.11}$$

We note that u^* and $\underline{u}^*$ are not quite optical functions of the metric. However, u^* can be seen as a sufficient approximation (more so than $u = t - r$, one optical function for the Minkowski metric). Additionally, we can define the optical weights

$$\tau_+^2 = (1 + \underline{u}^{*2}) \qquad \tau_-^2 = (1 + u^{*2}) \qquad \tau_0 = \tau_-/\tau_+. \tag{1.12}$$

Given $\partial_r = \omega^i \partial_i$ and $\overline{\partial}_i = \partial_i - \omega_i \partial_r$, we can define

$$\partial_{r^*} = \frac{1}{\partial_r(r^*)}\partial_r, \qquad \partial_{t^*} = \partial_{x^{*0}} = \partial_t - \partial_t(r^*)\partial_{r^*}, \qquad \partial_{x^{*i}} = \omega_i \partial_{r^*} + \frac{r}{r^*}\overline{\partial}_i. \tag{1.13}$$

We write

$$\overline{\partial}_{x^{*i}} = \frac{r}{r^*}\overline{\partial}_i,$$

a quantity which will often come up naturally later when calculating commutators.

This choice of fields lends itself to a natural null frame,

$$L^* = \partial_{t^*} + \partial_{r^*}, \qquad \underline{L}^* = \partial_{t^*} - \partial_{r^*}, \qquad S_i^* = \frac{r}{r^*} S_i, \tag{1.14}$$

where $S_i = \{S_1, S_2\}$ are piecewise defined fields forming an orthonormal frame tangent to the sphere (in the Minkowski metric). We note that our use of S_i^* is not strictly necessary, as they are of course proportional to S_i with a scalar factor close to 1; however, their use elucidates several cancellations which are necessary in handling Lie derivatives of the EM field, and which are not at all obvious using S_i alone. We use the shorthand

$$\mathcal{L} = \{L^*\}, \qquad \mathcal{T} = \{L^*, S_1^*, S_2^*\}, \qquad \mathcal{U} = \{L^*, \underline{L}^*, S_1^*, S_2^*\}. \tag{1.15}$$

This correspondence between our frames and fields have two advantages: First, they correspond well with estimates on the metric, for which one has the nice component estimate

$$|g_{\mathcal{L}\mathcal{T}}| \lesssim \frac{\epsilon \tau_-^{\gamma'}}{\tau_+^{1+\gamma'-\iota}},$$

where γ' and ι are defined as in (2.14) and (2.15). This makes it possible to achieve nicer estimates on the deformation tensor $\mathcal{L}_Z g$ without having to take into account the full geometric structure. This first comes into play in our energy estimates, where we must bound terms roughly behaving like

$$\int_0^T \int_{\Sigma_t} |(\mathcal{L}_{\overline{K_0^s}} g)(L^*, L^*)||D_{\underline{L}^*}\phi|^2 w \, dx \, dt,$$

where

$$\overline{K_0^s} = \frac{1}{2}(1 + \underline{u}^{*2s})L^* + \frac{1}{2}(1 + |u^*|^{2s})\underline{L}^*.$$

In order for this to be bounded by our energy, we need $\mathcal{L}_{\overline{K_0^s}} g(L^*, L^*)$ to be bounded roughly by $\epsilon \tau_-^{2s} \tau_+^{-1-\alpha}$, for some $\alpha > 0$. This estimate is in particular not possible when we replace L^* by its Minkowski analogue $\partial_t + \partial_r$ due to the behavior of the part of the metric like $M\chi/(1+r)\delta$. In the latter case, we would get decay like τ_+^{2s-2} which would lead to polynomial growth in the energy even in the (unfeasible) border case $s = 1/2$. It might be possible to mitigate this by subtracting off quantities close to the light cone, but this would be unintuitive without the geometric understanding of our modified fields.

Additionally, this null frame commutes well with the modified Lorentz fields used in [13], which again seem to be necessary in order to get the desired decay of metric terms. This follows from the fact that we would expect Lie derivatives of components of F to satisfy similar estimates to Lie derivatives of F. In the null frame in Minkowski, the best decay we would be able to expect for components like $|\mathcal{L}_Z^I F_{L^* S_j}|$ would be $\tau_+^{-2} \tau_-^{1/2-s}$, as opposed to the analogous term in [16], for which we would get the improved decay rate $\tau_+^{-3/2-s}$.

We contrast this to [15], which required less delicate peeling estimates, and consequently for which the standard null frame for Minkowski space sufficed.

We define the adapted null decomposition of F as follows:

$$\alpha_i = F_{L^* S_i^*} \qquad \underline{\alpha}_i = F_{\underline{L}^* S_i^*} \tag{1.16a}$$

$$\rho = \frac{1}{2} F_{\underline{L}^* L^*} \qquad \sigma = \frac{1}{2} F_{S_1^* S_2^*} \tag{1.16b}$$

In general we use the shorthand

$$|D_{S^*}\phi|^2 = |D_{S_1^*}\phi|^2 + |D_{S_2^*}\phi|^2 \qquad |\alpha|^2 = |\alpha_1|^2 + |\alpha_2|^2 \qquad |\underline{\alpha}|^2 = |\underline{\alpha}_1|^2 + |\underline{\alpha}_2|^2.$$

We additionally define the electromagnetic decomposition

$$E_i = F_{0i}, B_i = (*F)_{0i}, \tag{1.17}$$

where $*F$ is the Hodge dual of F. We can break E up into its divergence-free and curl-free components, E_{df} and E_{cf} respectively.

Before stating our result, we define the norms governing our initial conditions. For a $(0,k)$-tensor T, we define:

$$\|T\|^2_{H^{k,s_0}(\mathbb{R}^3)} = \sum_{|I|\le k} \sum_{\alpha_k \in (0,3)} \int_{\mathbb{R}^3} (1+r^2)^{s_0+|I|} |\underline{\nabla}^I T(\partial_{x^{*\alpha_1}}, \dots \partial_{x^{*\alpha_k}})|^2 \, dx, \tag{1.18}$$

where I is a multiindex. Here, $\underline{\nabla}$ and $\underline{D}$ are the covariant derivatives restricted to time slices.

Likewise, for a complex scalar field, we have the corresponding quantity

$$\|\phi\|^2_{H^{k,s_0}(\mathbb{R}^3)} = \sum_{|I|\le k} \int_{\mathbb{R}^3} (1+r^2)^{s_0+|I|} |\underline{D}^I \phi|^2 \, dx. \tag{1.19}$$

Theorem 1.1. *Take constants s_0, γ', δ such that $\frac{1}{2} < s_0 - \delta < \gamma' < 1 < s_0$, and ι such that the difference between any two of the previous quantities is at least 4ι. Additionally, take an integer $k_0 \ge 11$.*

There exists a constant $\epsilon_0 > 0$, with $\epsilon_0 \ll \iota$, such that if the metric satisfies (2.14) *and* (2.15) *for $\epsilon < \epsilon_0$, and if we take initial conditions $E_0, B_0, \phi_0, \dot{\phi}_0$ for F and ϕ satisfying*

$$\|E_{0df}\|_{H^{k_0,s_0}(\mathbb{R}^3)} + \|B_0\|_{H^{k_0,s_0}(\mathbb{R}^3)} + \|D\phi_0\|_{H^{k_0,s_0}(\mathbb{R}^3)} + \left\|\dot{\phi}_0\right\|_{H^{k_0,s_0}(\mathbb{R}^3)} < \epsilon \tag{1.20}$$

at time $t = 0$, then solutions to (1.2) *exist for all time and satisfy the bounds*

$$|\alpha| + \left|\frac{1}{r^*} D_{L^*}(r^*\phi)\right| \chi_{r^* > t/2+1/2} + |D_{L^*}\phi| \chi_{r^* < t/2+1/2} \lesssim \epsilon \tau_+^{-s-3/2}, \tag{1.21a}$$

$$|\underline{\alpha}| + |D_{\underline{L}^*}\phi| \lesssim \epsilon \tau_+^{-1} \tau_-^{-1/2-s}, \tag{1.21b}$$

$$|\sigma| + |\not{D}\phi| \lesssim \epsilon_0 \tau_+^{-1-s} \tau_-^{-1/2}, \tag{1.21c}$$

$$|\rho| \lesssim \epsilon \left(\tau_+^{-1-s} \tau_-^{-1/2} \chi_{r^* < t} + \tau_+^{-2} \chi_{r^* \ge t}\right), \tag{1.21d}$$

$$|\phi| \lesssim \epsilon \tau_+^{-1} \tau_-^{1/2-s}, \tag{1.21e}$$

for $s = s_0 - \delta$, and $\tau_\pm$ as defined in equation (1.12)*. Additionally, for all t, we have the energy bounds on* *the energy-momentum tensor*

$$\sum_{|I|\leq k}\left\|\mathcal{L}_X^I Q(\mathcal{U},\mathcal{U})(t,\cdot)\right\|_{L^2(x)} \lesssim \epsilon^2(1+t)^{-1}, \tag{1.22a}$$

$$\sum_{|I|\leq k}\left\|\mathcal{L}_X^I Q(\mathcal{U},\mathcal{U})(t,\cdot)\right\|_{L^1(x)} \lesssim \epsilon^2, \tag{1.22b}$$

where Q is defined as in (1.4).

The proof of this theorem structurally follows the space-time energy approach of [16], taking advantage of fractional Morawetz estimates used in that paper. The main conceptual difference is in the mass-corrected null frame and Lorenz fields. Additionally, there are significant error terms coming from the metric, which means that the required energy and commutator estimates are significantly more involved. Definitions and certain properties of these fields are outlined in section 2.

The last portion of section 2 includes several Morawetz estimates which provide motivation for the frameworks we use, including the fractional Morawetz estimate, the modified null frame, and the weight w. This estimate has at the end a discussion on certain issues adapting the full conformal Morawetz estimate to a general relativistic metric, and will hopefully cast light on our reasons for the fractional Morawetz estimate and the modified null frame and vector fields we use.

In sections 3 and 4, we first establish a fractional Morawetz estimate for the electromagnetic and scalar fields respectively. In the electromagnetic case, the proof is roughly straightforwardly adapted from [16], in which we consider the fractional acceleration field

$$\overline{K_0^s} = \frac{1}{2}\left((1+\underline{u}^{*2s})L^* + (1+|u^*|^{2s})\underline{L}^*\right)$$

for some constant $s \in (1/2, \gamma')$. The deformation tensor of this field satisfies certain positivity properties which were shown in [16], with error terms coming from the metric which we bound straightforwardly. Additionally, we subtract off a charge quantity $\overline{F}$ and analyzing the remainder tensor $F - \overline{F}$, as analyzing F on its own would result in field terms with insufficient decay in space.

For the scalar field, slightly more work is required, as the energy-momentum tensor is no longer trace-free. Therefore, we cannot rely on only the quasi-conformal killing structure of $\overline{K_0^s}$. We instead take a conformal transformation of the metric and apply the energy estimate to solutions of the wave equation on this new metric, then augment the resulting estimate using a weighted Poincaré-type inequality, loosely adapted from a similar estimate in [8].

Sections 5 and 6 establish L^∞ estimates on field quantities. These are conceptually straightforward weighted Klainerman-Sobolev estimates, with some additional care taken to account for the contribution of the charge. and issues establishing an energy norm along the light cone. Our estimates are Theorems 5.6 and 6.6.

In sections 7 and 8 we bound commutator terms coming from taking the energy estimate on Lie and complex covariant derivatives of F and ϕ respectively. This is achieved through a combination of the bilinear estimates used in [16] and a set of energies defined on the metric, which are bounded for a class of small-data solutions to the Einstein vacuum equations. In section 7.1 we also show that certain norms relating to initial data are equivalent or bounded by the initial data norms used in the main theorem.

Section 9 ties everything together: this will in particular establish that the right hand side of the earlier energy estimates can be easily bounded by the left hand side times a constant scaling with the size of the initial data. It follows that, for sufficiently small initial data, the energy is bounded. In particular, all parts of Theorem 1.1 follow directly from Theorem 9.1.

Section 10 is an appendix which contains some weighted Poincare- and Sobolev-type estimates which are of use in our proof.

1.1 Comparison to Previous Works

This work can be seen as a expansion on results found by Lindblad and Sterbenz in [16], who establish analogous estimates in Minkowski space. It is also worth mentioning results by [2] in which they establish similar results for Minkowski space in a way that can more readily be generalized into a gauge-free geometric setting, and which could potentially provide better understanding of the precise asymptotic behavior in a relativistic metric. Further analysis by Shiwu Yang in [24] and [25] has expanded on these results, showing stability even when the electromagnetic field F has large initial data. This approach uses the r-weighted energy decay method of [6] instead of the Morawetz estimate. This energy method was generalized to a broader class of metrics in [18], and was refined in [21].

From the relativistic viewpoint, I must first mention the landmark work of Christodoulou and Klainerman in [5], which established stability of the Minkowski spacetime solution to the Einstein equations, along with the dissertation of Zipser, [26], which uses their framework to establish stability results for the Einstein-Maxwell system. However, my analysis more closely follows the analagous result of Lindblad and Rodnianski in [15], [14] in which the authors establish stability in the harmonic gauge, as well as subsequent works by Loizelet and Speck in [17] and [22]. The dissertation of Loizelet extends the result of Lindblad and Rodnianski to solutions of the Einstein-Maxwell system in harmonic coordinates and Lorenz gauge, which unfortunately does not generalize well to a charged scalar field.

The analysis of Speck more closely follows our methods, in that he also looks only at the physical quantity F instead of A. However, this analysis does not use our modified null frame, which seems to be necessary in order to establish the necessary L^∞ estimates for the charged scalar field. This correction was originally inspired by considering exterior behavior for a conformal Morawetz estimate on the Schwarzschild metric carried out by Blue and Sterbenz in [3]. Similar work concerning Morawetz estimates on Schwarzschild for Maxwell's equations have been carried out by Anderson and Blue in [1] and Sterbenz and Tataru in [23], as well as for certain quasilinear equations by Lindblad and Tohaneanu in [12]. Additionally, we mention the use of a fractional Morawetz estimate in a metric which was used by Lindblad and Schlue in [11].

The modified null frame and fields have also been used independently by Oliver in [20], and later by Sterbenz and Oliver in [19]. The authors assume weaker conditions than we have here, only assuming boundedness of certain norms corresponding to certain perturbations of Minkowski space, instead of smallness, and derive certain estimates on solutions of linear and some nonlinear wave equations in these backgrounds.

In the sense of analysis of the Charge-Scalar Field system in Minkowski space, in addition to the papers by Lindblad-Sterbenz and Bieri-Miao-Shashahani ([16] and [2] respectively), I mention an earlier paper by Klainerman and Machedon, [10], which establishes existence and uniqueness for solutions using the Coulomb gauge, and one by Eardley and Moncrief, [7]. However, these papers offer little insight on asymptotics.

2 Notation and Preliminary Identities

2.1 General Notation

We mention notation regarding the metric here. As usual, we use Einstein summation notation, and the Greek indices α, β, etc. take values from 0 to 3 and are raised and lowered according to the metric g, with several exceptions, which we outline as follows:

First, in the estimate in section 2.6, everything is raised and lowered with respect to the Minkowski metric.

Additionally, the metric $m^{*\alpha\beta}$ is the inverse metric of m^*.

The last major exception is the analysis leading up to the first L^2 estimate in Section 4, which we will discuss more in-depth when we get to it. In particular, we set up a metric that preserves the geometric structure of our modified null fields while avoiding issues with singular behavior at the origin.

Given a covector ω_α and a vector X with components X^α, we say

$$\omega_X = \omega(X) = X^\alpha \omega_\alpha.$$

We can extend this to $(k,0)$-tensors. Note that we use F_{XY} and $F(X,Y)$ interchangeably. Given a frame $\{X_i\}$, and a vector Y^α, we write the raised decomposition

$$Y^\alpha = \sum_i Y^{X_i} X_i^\alpha.$$

We can decompose in more than one or all indices using

$$T^{\alpha\beta} = \sum_i T^{X_i \beta} X_i^\alpha.$$

One important consequence of this is that we have the identity

$$U^{\alpha\beta} V_{\alpha\beta} = \sum_{i,j} U^{X_i X_j} V_{X_i X_j},$$

etc. We have the rough identifications $F^{L^* S_j^*} + \frac{1}{2} F_{\underline{L}^* S_j^*} = O(|h|)|F|$, etc.

In order to avoid confusion, we will try to avoid the upper frame decomposition wherever possible; however, in some cases, such as calculations on the contraction $F_{\alpha\beta}F^{\alpha\beta}$, it is difficult to avoid.

Unless otherwise specified, English indices i, j, etc. range either from 1 to 3 or, with some abuse of notation, from 1 to 2 when we add quantities involving vectors S_i^* which are tangent to the sphere.

English indices are generally raised and lowered according to the Euclidean metric; this most naturally comes up when we say

$$\omega^i = \omega_i.$$

We take $a \lesssim b$ to mean

$$a \leq C_{s,\delta,\gamma',\iota,k} b,$$

, and similarly, $a \approx b$ to mean

$$C_{s,\delta,\gamma',\iota,k}^{-1} b \leq a \leq C_{s,\delta,\gamma',\iota,k} b,$$

where s, δ, and ι are parameters corresponding to weights on our scalar fields (we can ignore ι by defining it in terms of s), γ' governs the decay of the metric, and k is the maximum number of derivatives we work with. Intuitively, the constant C depends on the various decay rates of our quantities but not on the size of our initial conditions or the deviation of our metric from Minkowski, other than requiring that it falls under some fixed threshold. We remark that this smallness requirement naturally comes up when we compare to [13], as we have $\gamma' < \gamma - C\epsilon$, where ϵ is a bound for the energy on the metric.

In general, L^p norms taken without explicit domain will denote the spacetime norm

$$\|\phi\|_p = \|\phi\|_{L^p([0,T]\times\mathbb{R}^3)} .$$

L^p norms on other domains will be unambiguously denoted. In general, time-slice norms are implied to be in the time range $t \in [0,T]$ when we need to bound these quantities by our energy norms.

2.2 Vector Fields

The modified Lorentz fields can be defined as follows:

$$\mathbb{L} = \{\partial_{x^{*i}}, \quad \Omega^*_{ij} = x^{*i}\partial_{x^{*j}} - x^{*j}\partial_{x^{*i}} = \Omega_{ij}, \quad \Omega^*_{0j} = t^*\partial_{x^{*j}} + x^{*j}\partial_{t^*}, \quad Z^* = t^*\partial_{t^*} + x^{*i}\partial_{x^{*i}}\}. \tag{2.1}$$

We can define all other possible values of $\Omega^*_{\alpha\beta}$ by assuming $\Omega^*_{\alpha\beta} = -\Omega^*_{\beta\alpha}$. Given this set of fields, we can define the constants

$$c_X = \begin{cases} 2 & X = Z^*, \\ 0 & X = \partial_{x^{*\alpha}} \text{ or } X = \Omega^*_{\alpha\beta}. \end{cases} \tag{2.2}$$

These constants correspond with the conformal Killing and Killing behavior of the analogous Lorentz fields in Minkowski space. For motivation, we can say for now that the quantity $\mathcal{L}_X g - c_X g$ is small but nonzero (in a sense that we will clarify shortly).

Additionally, we have the radial Lorentz boost field

$$\Omega^*_{0r} = t^*\partial_{r^*} + r^*\partial_{t^*} = \frac{1}{2}\left(\underline{u}^*(\partial_{t^*} + \partial_{r^*}) - u^*(\partial_{t^*} - \partial_{r^*})\right).$$

Using this we can write

$$\Omega^*_{0j} = \omega_j\Omega^*_{0r} + t\overline{\partial}_{x^{*i}}, \tag{2.3}$$

which will simplify later commutator estimates.

We can now look at commutators, with the note that they generally behave identically to their unmodified equivalents. First, the commutators of two Lorentz fields:

$$[\partial_{x^{*\alpha}}, \partial_{x^{*\beta}}] = 0, \qquad [\partial_{x^{*\alpha}}, \Omega^*_{ij}] = \delta_{\alpha[i}\partial_{x^{*j}]}, \qquad [\partial_{x^{*\alpha}}, Z^*] = \partial_{x^{*\alpha}} \tag{2.4a}$$

$$[Z^*, \Omega^*_{\alpha\beta}] = 0, \qquad [\partial_{x^{*\alpha}}, \Omega^*_{0\beta}] = \delta_{\alpha(0}\partial_{x^{*\beta})}, \qquad [\Omega^*_{0i}, \Omega^*_{jk}] = \delta_{ij}\Omega^*_{0k} - \delta_{ik}\Omega^*_{0j} \tag{2.4b}$$

$$[\Omega^*_{0i}, \Omega^*_{0j}] = \Omega^*_{ij}, \qquad [\Omega^*_{ij}, \Omega^*_{kl}] = -\delta_{ik}\Omega^*_{jl} + \delta_{il}\Omega^*_{jk} + \delta_{jk}\Omega^*_{il} - \delta_{jl}\Omega^*_{ik}. \tag{2.4c}$$

Importantly, the commutator of any two of our modified Lorentz fields is a sum of modified Lorentz fields with constant coefficients. Now we look at the commutators between Lorentz fields and our modified null frame:

$$[L^*, \partial_{t^*}] = 0, \qquad [\underline{L}^*, \partial_{t^*}] = 0, \qquad [S^*_j, \partial_{t^*}] = 0, \tag{2.5a}$$

$$[L^*, \partial_{x^{*i}}] = -\frac{1}{r^*}\overline{\partial}_{x^{*i}}, \qquad [\underline{L}^*, \partial_{x^{*i}}] = \frac{1}{r^*}\overline{\partial}^*_i, \qquad [S^*_j, \partial_{x^{*i}}] = \frac{1}{r^*}a^k_{ij}(\omega)S^*_k + S^*_j(\omega_i)\partial_{r^*}, \tag{2.5b}$$

$$[L^*, Z^*] = L^*, \qquad [\underline{L}^*, Z^*] = \underline{L}^*, \qquad [S^*_j, Z^*] = S^*_j, \tag{2.5c}$$

$$[L^*, \Omega^*_{ij}] = 0, \qquad [\underline{L}^*, \Omega^*_{ij}] = 0, \qquad [S^*_k, \Omega^*_{ij}] = b^l_{ijk}(\omega)S^*_l, \tag{2.5d}$$

$$[L^*, \Omega^*_{0i}] = \omega_i L^* + \frac{r^* - t}{r^*}\overline{\partial}_{x^{*i}}, \; [\underline{L}^*, \Omega^*_{0i}] = -\omega_i\underline{L}^* + \frac{r^* + t}{r^*}\overline{\partial}_{x^{*i}}, \; [S^*_j, \Omega^*_{0i}] = S^*_j(\omega_i)\Omega^*_{0r} + \frac{t}{r^*}c^l_{ij}(\omega)S^*_l. \tag{2.5e}$$

Here, a, b, and c satisfy the conditions

$$a^1_{i1} = a^2_{i2} = b^1_{ij1} = b^2_{ij2} = c^1_{i1} = c^2_{i2} = 0, \tag{2.6a}$$

$$a^2_{i1} + a^1_{i2} = b^2_{ij1} + b^1_{ij2} = c^2_{i1} + c^1_{i2} = 0. \tag{2.6b}$$

We recall the Lie derivative formulas for one- and two-forms respectively:

$$(\mathcal{L}_X\omega)_Y = X(\omega_Y) - \omega([X, Y]), \tag{2.7}$$

$$(\mathcal{L}_X F)_{YZ} = X(F_{YZ}) - F([X, Y], Z) - F(Y, [X, Z]). \tag{2.8}$$

Remark. *One useful conequence is that when looking at "worse" components of Lie derivatives of "nicer" components in the extended exterior, these components are multiplied by a scalar function decreasing like (or* *faster than)* $\frac{\tau_-}{\tau_+}$*; i.e., we get better decay along the light cone. For instance, for a one-form* T_α*, we have*

$$(\mathcal{L}_{\Omega^*_{0i}}T)_{L^*} = \Omega^*_{0i}(T_{L^*}) - \omega_i T_{L^*} + \frac{u^*}{r^*}T_{\overline{\partial}_{x^{*i}}}.$$

We can write this in the following way, which will be useful when looking at the L^∞ estimates. We first define the following classes of functions: for any integer $k \geq 0$, we say a function ψ is in $\Psi_u^{K,N}$ if it can be written in the form

$$\sum_{\substack{k+l+m\leq n\leq N\\ l+m\leq n-K}} f_{k,l,m,n}(\omega)\frac{u^{*k}\underline{u}^{*l}t^m}{r^{*n}}. \tag{2.9}$$

Intuitively, this means that the function is bounded as you go to spatial infinity, with decay like τ_+^{-K} along the light cone. This comes into play when we take our null decomposition, so we can disregard the behavior in the spatial interior. Additionally, N is only a limiting constant, so we can disregard it in our geometric interpretation.

Additionally, the following is true:

Lemma 2.1. *For all vector fields* $X \in \mathbb{L}$*, and for any functions* $\psi \in \Psi_u^{K,N}$*, it follows that* $X\psi \in \Psi_u^{K,N+1}$*. Additionally, if* $f_1 \in \Psi_u^{K_1,N_1}$ *and* $f_2 \in \Psi_u^{K_2,N_2}$*, then* $f_1 f_2 \in \Psi_u^{K_1+K_2,N_1+N_2}$

Proof. This is straightforward but tedious to prove. If $X = \partial_{x^{*\alpha}}$, it is easy to see that differentiating lowers the power of k, l or m by one and multiplying by an angular function (which may be identically 0) if the derivative lands on u^*, $\underline{u}^*$ or t respectively, and increases n by 1, multiplying by an angular function again, if the derivative lands on f or r^*. Other derivatives behave similarly, and will be left as an exercise to the reader.

The product relation is easy to prove, and follows from expanding the product and verifying that the product terms satisfy the resulting bounds. $\square$

Given this, and the Lie derivative formula (2.8), we have the following lemma:

Lemma 2.2. *For a given 2-form* F*, a field* $X \in \mathbb{L}$*, a function* g *in* $\Psi_u^{K,N}$*, and the corresponding null decomposition of* F $\{\alpha, \underline{\alpha}, \rho, \sigma\}$*, we can rewrite*

$$g\alpha_i[\mathcal{L}_X F] - X(g\alpha_i[F]) = f_1^j\alpha_j[F] + f_2\rho[F] + f_3\sigma[F], \tag{2.10a}$$

where $f_1 \in \Psi_u^{K,N+1}$, *and* $f_2, f_3 \in \Psi_u^{K+1,N+1}$. *Additionally,*

$$g\rho[\mathcal{L}_X F] - X(g\rho[F]) = f_1^j \alpha_j[F] + f_2\rho[F] + f_3\sigma[F] + f_4^j \underline{\alpha}_j[F], \tag{2.10b}$$

$$g\sigma[\mathcal{L}_X F] - X(g\sigma[F]) = f_1^j \alpha_j[F] + f_2\rho[F] + f_3\sigma[F] + f_4^j \underline{\alpha}_j[F], \tag{2.10c}$$

where $f_1, f_2, f_3 \in \Psi_u^{K,N+1}$, *and* $f_4 \in \Psi_u^{K+1,N+1}$, *and*

$$g\underline{\alpha}_i[\mathcal{L}_X F] - X(g\underline{\alpha}_i[F]) = f_1^j \alpha_j[F] + f_2\rho[F] + f_3\sigma[F] + f_4^j \underline{\alpha}_j[F], \tag{2.10d}$$

where $f_1, f_2, f_3, f_4 \in \Psi_u^{K,N+1}$.

Proof. These follow straightforwardly from (2.5). We prove the first estimate here, others are similar: We first have the expansion

$$\begin{aligned}\alpha_i[g\mathcal{L}_X F] &= gX(\alpha_i[F]) - gF([X, L^*], S_i^*) - gF(L^*, [X, S_i^*]) \\ &= X(g\alpha_i[F]) - X(g)\alpha_i[F] - gF([X, L^*], S_i^*) - gF(L^*, [X, S_i^*]).\end{aligned}$$

The first term is subtracted off. The second term is treated using properties of $X(g)$ coming from the first part of Lemma 2.1. The other terms are slightly more involved and use equation (2.5) and the product properties in Lemma 2.1.

We have from (2.5) that $[X, L^*] = f_1 L^* + f_2^i S_i^*$, where $f_1 \in \Psi_u^{0,0}$ and $f_2^i \in \Psi_u^{1,1}$. Likewise, $[X, S_i^*] = f_1 L^* + f_2^i S_i^* + f_3 \underline{L}^*$, where $f_1, f_2^i \in \Psi_u^{0,1}$ and $f_3 \in \Psi_u^{1,1}$. Applying the product relation in Lemma 2.1 gives us the desired properties. □

This formula has as a consequence nice inductive properties for higher numbers of Lie derivatives.

Corollary 2.3. *Given a two-form* F, *and* $\tau_0 = \tau_-/\tau_+$, *we have that if* X^I *is a multiindex of fields in* $\mathbb{L}$, *it* *follows that in the region $r^* \geq t/2 + 1/2$,*

$$X^I(\alpha_i[F]) \lesssim \sum_{\substack{|J|\leq|I| \\ Y\in\mathbb{L}}} |\alpha[\mathcal{L}_Y^J F]| + |\tau_0\rho[\mathcal{L}_Y^J F]| + |\tau_0\sigma[\mathcal{L}_Y^J F]| + \tau_0^2|\underline{\alpha}[\mathcal{L}_Y^J F]|, \tag{2.11a}$$

$$X^I(\rho[F]) \lesssim \sum_{\substack{|J|\leq|I| \\ Y\in\mathbb{L}}} |\alpha[\mathcal{L}_Y^J F]| + |\rho[\mathcal{L}_Y^J F]| + |\sigma[\mathcal{L}_Y^J F]| + \tau_0|\underline{\alpha}[\mathcal{L}_Y^J F]|, \tag{2.11b}$$

$$X^I(\sigma[F]) \lesssim \sum_{\substack{|J|\leq|I| \\ Y\in\mathbb{L}}} |\alpha[\mathcal{L}_Y^J F]| + |\rho[\mathcal{L}_Y^J F]| + |\sigma[\mathcal{L}_Y^J F]| + \tau_0|\underline{\alpha}[\mathcal{L}_Y^J F]|, \tag{2.11c}$$

$$X^I(\underline{\alpha}_i[F]) \lesssim \sum_{\substack{|J|\leq|I| \\ Y\in\mathbb{L}}} |\alpha[\mathcal{L}_Y^J F]| + |\rho[\mathcal{L}_Y^J F]| + |\sigma[\mathcal{L}_Y^J F]| + |\underline{\alpha}[\mathcal{L}_Y^J F]|. \tag{2.11d}$$

Proof. This follows from repeated iteration of Lemma 2.2 from the inside out, combined with the estimate that if ψ is contained in $\Psi_j^{K,N}$, it satisfies the estimate

$$\psi \lesssim_N \frac{(1+u^*)^K}{r^{*K}} \lesssim_N \tau_0^K$$

□

whenever $r^* \geq t/2 + 1/2$.

Next, we discuss a lemma which will be useful when commuting derivatives with Lie derivatives:

Lemma 2.4. *For all vector fields $X \in \mathbb{L}$, the following estimates hold for all multiindices α:*

$$\partial_\alpha^I X^\beta \lesssim_{|I|} \begin{cases} 1, & |I| = 1, \\ M\ln(\tau_+){\tau_+}^{-|I|}, & |I| \geq 2. \end{cases} \tag{2.12}$$

2.3 Assumptions on the Metric

One advantage of the null frame of the previous section over the standard null frame in Minkowski space is that we can essentially combine the Minkowski and ADM parts of the metric (m and h_0 respectively in [15]).

We split $g_{\alpha\beta} = \tilde{m}_{\alpha\beta} + h_{\alpha\beta}$, $g^{\alpha\beta} = (\tilde{m}^{-1})^{\alpha\beta} + H^{\alpha\beta}$, where

$$\tilde{m}_{\alpha\beta} = m_{\alpha\beta} + \frac{M\chi}{1+r}\delta_{\alpha\beta}. \tag{2.13}$$

with all other components equal to 0. Note that $(\tilde{m}^{-1})$ here is the inverse metric of $\tilde{m}$. This use is identical to that in [15]. We note that

$$\tilde{m}_{L^*L^*}, \tilde{m}_{L^*S_j^*} \approx M\tau_+^{-2}$$

everywhere except in the far interior (in particular, the region where $\partial\chi \neq 0$). In the far interior, $\tilde{m}_{L^*L^*}$ decays like $M\tau_+^{-1+\iota}$, which we can in particular rewrite as $M\tau_-^{1+\iota}\tau_+^{-2}$. We can consider the latter as a uniform bound.

Now we look at H. Since we are for now treating this as a model estimate for the full EMKG system, we assume L^∞ estimates on low derivatives of H and L^2 estimates on high derivatives of H as follows: We first consider $\gamma' < \gamma < 1$. Then, we define the class $G_{\epsilon,\gamma',\gamma,k}$ to be the set of metrics g with $M \leq \epsilon$ (where M is the multiplier corresponding to the ADM part of the metric) satisfying the L^∞ norms

$$M < \epsilon, \tag{2.14a}$$

$$|\mathcal{L}_{\mathbb{L}}^I h| < \epsilon\tau_+^{-1+\iota}, \tag{2.14b}$$

$$|(\mathcal{L}_{\mathbb{L}}^I h)_{\mathcal{L}\mathcal{T}}| < \epsilon\tau_-^{\gamma'}\tau_+^{-\gamma'-1+\iota}, \tag{2.14c}$$

for a multiindex $I, |I| \leq k-6$, and $X \in \mathbb{L}$, and

$$\left\|\tau_+^{-1/2-\iota}|\partial\mathcal{L}_X^I h|w_1^{1/2}\right\|_{L^2(t,x)} + \left\|\tau_+^{-1/2-\iota}\tau_-^{-1}|\mathcal{L}_X^I h|w_1^{1/2}\right\|_{L^2(t,x)} \leq \epsilon, \tag{2.15a}$$

$$\left\|\tau_-^{-1/2}\tau_+^{-\iota}|(\partial\mathcal{L}_X^I h)_{\mathcal{L}\mathcal{L}}| + |\overline{\partial}\mathcal{L}_X^I h|)w_1^{1/2}\right\|_{L^2(t,x)} + \left\|\tau_-^{-3/2}\tau_+^{-\iota}|(\mathcal{L}_X^I h)_{\mathcal{L}\mathcal{L}}|w_1^{1/2}\right\|_{L^2(t,x)} \leq \epsilon, \tag{2.15b}$$

for $|I| \leq k$, some small constant $\iota > 0$,and

$$w_1 = \begin{cases} 1 & r^* \leq t, \\ 1 + (r^* - t) & r^* \geq t. \end{cases} \tag{2.16}$$

In general, $k \geq 11$, and $1/2 + 4\iota < \gamma' < 1 - 4\iota$. Finally, we assume the initial conditions that at time $t = 0$, we have a split metric, with

$$g_{00} = -\left(1 - \frac{M\chi}{1+r}\right), \qquad g_{0i} = 0 \tag{2.17}$$

In the spacetime integrals the value of τ_- on the left is that corresponding to the integrated time variable

τ. These are consistent with estimates established in [15] and [13], where we use the weak null condition for the spacetime estimates on the $\mathcal{L}\mathcal{L}$ terms. These will in particular be useful when taking commutators.

We have analogous estimates on components of the raised metric. In particular, if we write out the matrix $g_{X_iX_j}$, where X_i, X_j are elements in our null frame, and then take the inverse via the adjoint method, we see that every term appearing in $g^{\underline{L}^*\underline{L}^*}, g^{\underline{L}^*S^*}$ contains either a term decaying like $\tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}$ or $\tau_+^{-2+2\iota}$. We therefore have the estimates

$$|g^{\underline{L}^*\underline{L}^*}| + |g^{\underline{L}^*S^*}| \lesssim \epsilon\tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}, \tag{2.18a}$$

$$\left|g^{L^*\underline{L}^*} + \frac{1}{2}\right| + \left|g^{S_i^*S_i^*} - 1\right| \lesssim \epsilon\tau_+^{-1+\iota}, \tag{2.18b}$$

$$|g^{L^*L^*}| + |g^{S_1^*S_2^*}| + |g^{L^*S^*}| \lesssim \epsilon\tau_+^{-1+\iota}. \tag{2.18c}$$

We can therefore prove the following lemma:

Lemma 2.5. *Given a two-tensor $T_{\alpha\beta}$, we can define the following norms:*

$$|T| = |T_{\mathcal{U}\mathcal{U}}|, |\not{T}| = |T_{\mathcal{T}\mathcal{T}}| + |T_{\mathcal{L}\mathcal{U}}| + |T_{\mathcal{U}\mathcal{L}}|, \tag{2.19}$$

where we recall $\mathcal{L} = \{L^*\}, \mathcal{T} = \{L^*, S^*\}, \mathcal{U} = \{L^*, \underline{L}^*, S^*\}$ and each norm denotes the sum of the norms for each field in the sets $\mathcal{L}, \mathcal{U}, \mathcal{T}$. Then, we have the following estimates on the raised components T^{XY}:

$$\left|T^{\underline{L}^*\underline{L}^*} - \frac{1}{4}T_{L^*L^*}\right| \lesssim \epsilon\left(\tau_+^{-1+\iota}|T_{\mathcal{L}\mathcal{L}}| + \tau_-^{\gamma}\tau_+^{-1-\gamma'+\iota}|\not{T}| + \tau_-^{2\gamma'}\tau_+^{-2-2\gamma'+2\iota}|T|\right), \tag{2.20a}$$

$$\left|T^{\underline{L}^*S_j^*} + \frac{1}{2}T_{L^*S_j^*}\right| + \left|T^{S_j^*\underline{L}^*} + \frac{1}{2}T_{S_j^*L^*}\right| \lesssim \epsilon\left(\tau_+^{-1+\iota}(|T_{\mathcal{L}\mathcal{T}}| + |T_{\mathcal{T}\mathcal{L}}|) + \tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}|T|\right), \tag{2.20b}$$

$$\left|T^{\underline{L}^*L^*} - \frac{1}{4}T_{L^*\underline{L}^*}\right| + \left|T^{L^*\underline{L}^*} - \frac{1}{4}T_{\underline{L}^*L^*}\right| \lesssim \epsilon\left(\tau_+^{-1+\iota}|\not{T}| + \tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}|T|\right), \tag{2.20c}$$

$$\left|T^{S_i^*S_j^*} - T_{S_i^*S_j^*}\right| \lesssim \epsilon\left(\tau_+^{-1+\iota}|\not{T}| + \tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}|T|\right), \tag{2.20d}$$

$$\left|T^{L^*S_j^*} + \frac{1}{2}T_{\underline{L}^*S_j^*}\right| + \left|T^{S_j^*L^*} + \frac{1}{2}T_{S_j^*\underline{L}^*}\right| \lesssim \epsilon\tau_+^{-1+\iota}|T|, \tag{2.20e}$$

$$\left|T^{L^*L^*} - \frac{1}{4}T_{\underline{L}^*\underline{L}^*}\right| \lesssim \epsilon\tau_+^{-1+\iota}|T| \tag{2.20f}$$

Proof. This almost entirely follows from equation (2.18). As an example, we prove our result for the first term on the left hand side of equation (2.20b). We first rewrite

$$T^{\underline{L}^* S_j^*} = \sum_{X,Y \in \mathcal{U}} g^{\underline{L}^* X} T_{XY} g^{Y S_j^*} \tag{2.21}$$

If $X = L^*$ and $Y = S_j^*$, we have the term which is subtracted off plus an error term which is directly bounded by the first term on the right hand side. If $X = L^*$ and $Y = L^*$ or S_i^*, we likewise have something bounded by the first term on the right. If $X = L^*$ and $Y = \underline{L}^*$, we have a term like $g^{\underline{L}^* S_j^*}$, which has our better decay norm. If $X \neq L^*$, we likewise have either $g^{\underline{L}^* \underline{L}^*}$ or $g^{\underline{L}^* S_i^*}$, both of which have our better decay norms. Other component estimates follow similarly. ☐

Now we define a vector which will be useful in the energy estimate:

$$\tilde{L^*} = \nabla u^*, \tag{2.22}$$

the normal vector to surfaces of constant u^*. We note that $\tilde{L^*}$ is very close to a scalar multiple of L, in the sense that

$$g_{\tilde{L^*} L^*}, g_{\tilde{L^*} S_j^*} = 0,$$

which is a restatement of the relations $L^*(u^*), S_j^*(u^*) = 0$. We can say in particular that

$$\tilde{L^*}^{\underline{L}^*} \lesssim \epsilon \tau_-^{\gamma'} \tau_+^{-1-\gamma'}.$$

This will be useful later, as we can bound error terms from the metric using an integrated L^∞ estimate, at the expense of requiring more derivatives of the function in the energy.

2.4 Lie Derivatives and Commutators

We recall the definition of the deformation tensor

$$^{(X)}\pi = \mathcal{L}_X g = 2 \cdot \mathrm{symm}(\nabla X), \tag{2.23}$$

where the last identity is a straightforward calculation. It follows that if X is Killing or conformal Killing, $^{(X)}\pi$ is 0 or a scalar multiple of the metric respectively. In our work, we cannot assume any Killing or conformal Killing fields. However, as we have discussed, our modified Lorentz fields $\mathbb{L}$ behave similarly to their analogues in Minkowski space, in the sense that error terms coming from quantities which are 0 in the Minkowski metric satisfy certain bounds in our spacetimes.

We first take a notational tool defined for the Lorentz fields $\mathbb{L}$, $\widetilde{\mathcal{L}}_X$, such that

$$\widetilde{\mathcal{L}}_X(T^{\alpha\beta}) = (\mathcal{L}_X T)^{\alpha\beta} + c_X T^{\alpha\beta}, \tag{2.24a}$$

$$\widetilde{\mathcal{L}}_X(T_{\alpha\beta}) = (\mathcal{L}_X T)_{\alpha\beta} - c_X T_{\alpha\beta}, \tag{2.24b}$$

where the c_X are the Killing coefficients defined in (2.2). The primary advantage of this is that we can easily reduce deformation tensors into their error terms. In each case, we define the iterated reduced deformation tensors

$$({}^{(X^I)}\tilde{\pi})_{\alpha\beta} = (\widetilde{\mathcal{L}}_X^I g)_{\alpha\beta}, \tag{2.25a}$$

$$({}^{(X^I)}\tilde{\pi}^\dagger)^{\alpha\beta} = (\widetilde{\mathcal{L}}_X^I g^\dagger)^{\alpha\beta}. \tag{2.25b}$$

These are set up so that whenever X is a Lorentz field, ${}^{(X)}\tilde{\pi}$ and ${}^{(X)}\tilde{\pi}^\dagger$ satisfy the same L^2 and L^∞ norms as h, H, the deviations of the metric g from our model metric $\tilde{m}$. Additionally, for a single vector field X, we have the formula

$$ {}^{(X)}\tilde{\pi}^{\alpha\beta} = -{}^{(X)}\tilde{\pi}^{\dagger\,\alpha\beta}, $$

where indices on the left hand side are raised with respect to g.

Additionally, we have the useful estimate

$$\left|Y\left(\mathcal{L}_X^I(\nabla\cdot X_1)\right)\right| \lesssim \sum_{\substack{|I_1|+|I_2|\leq|I|+1 \\ X\in\mathbb{L}}} \left|Y\left(g^{\alpha\beta}\,{}^{(X^{I_2})}\tilde{\pi}_{\alpha\beta}\right)\right| + \left|Y\left({}^{(X^{I_1})}\tilde{\pi}^{\dagger\,\alpha\beta}\,{}^{(X^{I_2})}\tilde{\pi}_{\alpha\beta}\right)\right|. \tag{2.26}$$

This follows from taking the trace of the identity (2.23), subtracting off the constant term $c_X \mathrm{tr}(g)$, and expanding the Lie derivatives using (2.24).

These quantities are of course 0 in the Minkowski case, as these vectors have constant divergence. We now show that these quantities also satisfy analogous estimates to (2.14) and (2.15). These properties will be necessary when commuting derivatives through various operators.

Proposition 2.6. *Given the inequalities* (2.14), (2.15)*, and* X^I *a set of Lorentz fields, we can get the* *following analogous results on our reduced deformation tensors:*

$$|^{(X^I)}\tilde{\pi}| \lesssim \epsilon\tau_+^{-1+\iota}, \tag{2.27a}$$

$$|(^{(X^I)}\tilde{\pi}_{\mathcal{L}\mathcal{T}}| \lesssim \epsilon\tau_-^{\gamma'}\tau_+^{-\gamma'-1+\iota}, \tag{2.27b}$$

for $|I| \leq k-6$, *and*

$$\left\|\tau_+^{-1/2-\iota}|\partial^{(X^I)}\tilde{\pi}|w_1^{1/2}\right\|_{L^2(t,x)} + \left\|\tau_+^{-1/2-\iota}\tau_-^{-1}|^{(X^I)}\tilde{\pi}|w_1^{1/2}\right\|_{L^2(t,x)} \lesssim \epsilon, \tag{2.28a}$$

$$\left\|\tau_-^{-1/2}\tau_+^{-\iota}|(\partial^{(X^I)}\tilde{\pi})_{\mathcal{L}\mathcal{L}}| + |\overline{\partial}^{(X^I)}\tilde{\pi}|)w_1^{1/2}\right\|_{L^2(t,x)} + \left\|\tau_-^{-3/2}\tau_+^{-\iota}|(^{(X^I)}\tilde{\pi})_{\mathcal{L}\mathcal{L}}|w_1^{1/2}\right\|_{L^2(t,x)} \lesssim \epsilon, \tag{2.28b}$$

for $|I| \leq k$

Proof. We prove this for the lower indices; the upper indices follow straightforwardly. In all cases it suffices to prove this with all modified Lie derivatives after the first replaced with regular Lie derivatives, as we can expand $\widetilde{\mathcal{L}}_X$.

The L^∞ estimates are easy to show, as we can write the reduced deformation tensor as a sum of terms like g, h. For the L^2 estimates, if we expand everything out, It suffices to show that Lie derivatives of the $\tilde{m}$ terms satisfy the same estimates, which follows from expanding Lie derivatives and applying the identities (2.5). We detail these as follows:

First, we take the decomposition

$$\tilde{m}(L^*, L^*) = \left(-1 + \frac{M\chi}{1+r}\right) + \left(\frac{1-\partial_t(r^*)}{\partial_r(r^*)}\right)^2\left(1 + \frac{M\chi}{1+r}\right), \tag{2.29a}$$

$$\tilde{m}(\underline{L}^*, \underline{L}^*) = \left(-1 + \frac{M\chi}{1+r}\right) + \left(\frac{-1-\partial_t(r^*)}{\partial_r(r^*)}\right)^2\left(1 + \frac{M\chi}{1+r}\right), \tag{2.29b}$$

$$\tilde{m}(\underline{L}^*, L^*) = \tilde{m}(L^*, \underline{L}^*) = \left(-1 + \frac{M\chi}{1+r}\right) + \left(\frac{(1-\partial_t(r^*))^2}{(\partial_r(r^*))^2}\right)\left(1 + \frac{M\chi}{1+r}\right) \tag{2.29c}$$

$$\tilde{m}(S_i^*, S_i^*) = \left(\frac{r}{r^*}\right)^2\left(1 + \frac{M\chi}{1+r}\right), \tag{2.29d}$$

with other components equal to 0. These can be thought of as the corresponding constant in the Minkowski metric, plus terms with better decay (roughly like $M\tau_+^{-1}\ln(\tau_+)$ for the angular components and better for the other components).

For $X = \partial_{t^*}, Z^*, \Omega^*_{ij}$, we have the formula coming from (2.5) and the identity (2.8)

$$(\mathcal{L}_X \tilde{m})_{Y_1 Y_2} = X(\tilde{m}_{Y_2 Y_2}) + c_X \tilde{m}_{Y_1 Y_2},$$

i.e. $(\widetilde{\mathcal{L}}_X \tilde{m})(Y_1, Y_2) = X(\tilde{m}(Y_1, Y_2))$.

For $X = \partial_{x^{*i}}$, we have the same thing in all but the mixed components like $\tilde{m}(L^*, S^*_i)$, $\tilde{m}(\underline{L}^*, S^*_i)$. For the first, we have

$$(\mathcal{L}_{\partial_{x^{*i}}} \tilde{m})_{L^* S^*_j} = \frac{1}{r^*} \tilde{m}(\overline{\partial}_{x^{*i}}, S^*_j) - S^*_j(\omega_i) \tilde{m}(L^*, \partial_{r^*}). \tag{2.30}$$

With similar calculations for other terms. Noting the relation $S^*_j(\omega_i) = \frac{1}{r} S^{*i}_j$, where the S^{*i}_j denotes the ith component of S^*_j, we see that these cancel out up to order $M\tau_+^{-2} \ln(\tau_+)$, and are 0 in the interior.

The hardest component corresponds with the Lorentz boosts Ω^*_{0i}. We see here that the $\tilde{m}_{L^* L^*}$ component satisfies

$$\mathcal{L}_{\Omega^*_{0i}} \tilde{m}_{L^* L^*} = \Omega^*_{0i}(\tilde{m}_{L^* L^*}) - 2\omega_i \tilde{m}_{L^* L^*}$$

which satisfies our estimate, as well as

$$\mathcal{L}_{\Omega^*_{0i}} \tilde{m}_{L^* S^*_j} = \frac{r^* - t}{r^*} \tilde{m}(\overline{\partial}_{x^{*i}}, S^*_j) + S^*_j(\omega_i) \left(\underline{u}^*(\tilde{m}(L^*, L^*) + u^*(\tilde{m}(L^*, \underline{L}^*))) \right).$$

Importantly, the worst decay we have here is again $M\tau_- \tau_+^{-2} \ln(\tau_+)$.

First, we have the uniform estimate, for all $X \in \mathbb{L}$ $|I| \leq k$, $Y \in \{L^*, \underline{L}^*, S^*_j\}$:

$$|X^I(\tilde{m}_{Y_1 Y_2})| \lesssim M\tau_+^{-1} \ln(\tau_+), \qquad |\partial X^I(\tilde{m}_{Y_1 Y_2})| \lesssim M\tau_+^{-2} \ln(\tau_+).$$

We can get similar Lie derivative estimates without issue. Using the identity $\tau_-^{-1/2} w_1^{1/2} \lesssim 1$, we get decay of all terms containing $\partial X^I(\tilde{m}_{Y_1 Y_2})$ like $\tau_+^{-4-\epsilon_0}$ after squaring, for some $\epsilon_0 > 0$. The desired bound follows from integrating directly. It remains to show results for the undifferentiated terms, where we in particular we cannot use the Hardy estimate due to the lack of the weight. We instead integrate directly. In particular, we need to show that the quantity

$$\int_0^T \int_{\Sigma_t} \tau_+^{-3-\iota} \tau_-^{-1} \, dx \, dt \lesssim \epsilon.$$

This follows from first decomposing to regions of the form $n < \tau_- < n+1$, and noting that at each region, for fixed time t the volume of this region in three dimensions scales like τ_+^2. Since at the spatial origin and at time $t = 0$ we have $\tau_+ \approx \tau_-$, we can therefore estimate the integral over each region by

$$\int_{\tau_-}^{\infty} \tau_-^{-1}\tau_+^{-1-\iota}\,d\tau_+ \lesssim_\iota \tau_-^{-1-\iota}.$$

Summing these over n gives our desired inequality. Finally, we look at the estimate on the undifferentiated nice component. We recall the estimate

$$\tilde{m}_{L^*L^*} \lesssim M\ln(\tau_+)\tau_+^{-2},$$

which holds for all derivatives as well. We consider the commutator part of the Lie derivative which gives us

$$|(\mathcal{L}_X^I h)_{\mathcal{L}\mathcal{L}}| \lesssim M\tau_-\tau_+^{-2}\ln(\tau_+). \tag{2.31}$$

The final estimate follows from direct integration. □

Next, we look at standard and covariant derivatives commuted through the metric. First, for standard derivatives ∂_γ, and for tensors $T^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}$, we have that

$$\begin{aligned}[\partial_\gamma, \mathcal{L}_X]T = &-(\partial_\gamma\partial_\delta X^{\alpha_1})T^{\delta\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} - \ldots - (\partial_\gamma\partial_\delta X^{\alpha_m})T^{\alpha_1\alpha_2\ldots\delta}_{\beta_1\beta_2\ldots\beta_n} + \\ &+ (\partial_\gamma\partial_{\beta_1} X^{\delta})T^{\alpha_1\alpha_2\ldots\alpha_m}_{\delta\beta_2\ldots\beta_n} + \ldots + (\partial_\gamma\partial_{\beta_n} X^{\delta})T^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\delta}.\end{aligned} \tag{2.32}$$

We have an analogous result for the covariant derivative:

$$\begin{aligned}[\nabla_\gamma, \mathcal{L}_X]T = &-(\nabla_\gamma\nabla_\delta X^{\alpha_1})T^{\delta\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} - \ldots - (\nabla_\gamma\nabla_\delta X^{\alpha_m})T^{\alpha_1\alpha_2\ldots\delta}_{\beta_1\beta_2\ldots\beta_n} + \\ &+ (\nabla_\gamma\nabla_{\beta_1} X^{\delta})T^{\alpha_1\alpha_2\ldots\alpha_m}_{\delta\beta_2\ldots\beta_n} + \ldots + (\nabla_\gamma\nabla_{\beta_n} X^{\delta})T^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\delta}.\end{aligned} \tag{2.33}$$

In each case, if T is a scalar, the corresponding commutator is 0.

In the Minkowski case, these are again 0 whenever X is a Lorentz field, as X^α is constant or linear in the standard frame.

For all vector fields X and all antisymmetric $(2,0)$-tensors F, we have the identity

$$[\nabla_\beta, \mathcal{L}_X]F^{\alpha\beta} = -(\nabla_\delta\nabla_\beta X^\beta)F^{\alpha\delta}. \tag{2.34}$$

This is straightforward to prove:

$$[\nabla_\beta, \mathcal{L}_X]F^{\alpha\beta} = \nabla_\beta \left(X^\delta \nabla_\delta F^{\alpha\beta} - (\nabla_\delta X^\alpha)F^{\delta\beta} - (\nabla_\delta X^\beta)F^{\alpha\delta}\right) - X^\delta \nabla_\delta \nabla_\beta F^{\alpha\beta} + (\nabla_\delta X^\alpha)\nabla_\beta F^{\delta\beta} -$$
$$= X^\delta [\nabla_\beta, \nabla_\delta]F^{\alpha\beta} - (\nabla_\beta \nabla_\delta X^\alpha)F^{\delta\beta} - (\nabla_\beta \nabla_\delta X^\beta)F^{\alpha\delta}.$$

Expanding the first term using the Riemann curvature tensor, symmetrizing the derivatives in the middle term and commuting the derivatives in the last term, then taking advantage of the antisymmetry of F and the Bianchi identity

$$R^\alpha_{\beta\gamma\delta} + R^\alpha_{\gamma\delta\beta} + R^\alpha_{\delta\beta\gamma} = 0$$

gives us the desired identity.

Likewise, we can define the complex Lie derivative

$$\mathcal{L}^{\mathbb{C}}_X = \mathcal{L}_X + iA_X. \tag{2.35}$$

This can of course be seen as an analogue to the standard Lie derivative which works well with the complex covariant derivative. We can write the commutators

$$[D_\beta, \mathcal{L}^{\mathbb{C}}_X]\psi = iF_{\beta X}\psi, \tag{2.36a}$$
$$[D^\alpha, \mathcal{L}^{\mathbb{C}}_X]\psi = ig^{\alpha\beta}F_{\beta X}\psi + {}^{(X)}\pi^{\alpha\beta}D_\beta\psi, \tag{2.36b}$$
$$[D_\alpha, \mathcal{L}^{\mathbb{C}}_X]\eta^\alpha = iF_{\alpha\beta}\eta^\alpha X^\beta - (\nabla_\beta \nabla_\alpha X^\alpha)\eta^\beta. \tag{2.36c}$$

The first equation comes from expanding everything out, the second equation comes from contracting D with the inverse metric, and the third equation comes from the identities $[D_\alpha, D_\beta] = [\nabla_\alpha, \nabla_\beta] + iF_{\alpha\beta}$.

It follows that

$$[\Box^{\mathbb{C}}_g, D_X]\psi = iD^\beta(F_{\beta X}\psi) + D_\alpha({}^{(X)}\pi^{\alpha\beta}D_\beta\psi) + iF_{\alpha X}D^\alpha\psi - \nabla_\beta(\nabla \cdot X)D^\beta\psi \tag{2.37}$$

We now look into iterating these commutators. First, however, we mention an important L^∞ estimate: First, from equation (2.12), we know that

$$[\partial_\gamma, \mathcal{L}_X]T \lesssim_{\gamma'} M{\tau_+}^{-1-\gamma'}|T|. \tag{2.38}$$

Additionally, we know that $\nabla_\delta X^\alpha = \partial_\delta X^\alpha + \Gamma^\alpha_{\delta X}$, and likewise,

$$\nabla_\gamma \nabla_\delta X^\alpha = \partial_\gamma \left(\partial_\delta X^\alpha + \Gamma^\alpha_{\delta X}\right) + \Gamma^\beta_{\gamma\delta} \left(\partial_\beta X^\alpha + \Gamma^\alpha_{\beta X}\right) - \Gamma^\alpha_{\gamma\beta} \left(\partial_\delta X^\beta + \Gamma^\beta_{\delta X}\right).$$

We note that Γ satisfies the following abstract estimates: in the extended exterior, if $L^* = X_1$, $S^*_j = X_0$, $\underline{L}^* = X_{-1}$, then

$$\Gamma_{X_a X_b X_c} \lesssim \begin{cases} \epsilon \tau_+^{-1-\gamma'+\iota} \tau_-^{-1+\gamma'} & a+b+c \geq 0 \\ \epsilon \tau_+^{-1+\iota} \tau_-^{-1} & \text{otherwise.} \end{cases} \tag{2.39}$$

We can raise and lower indices according to the metric $\tilde{m}$ without jeopardizing our estimates. The best that we can assume here without component estimates is

$$|\nabla^2 X| \lesssim \epsilon \tau_+^{-\gamma'}.$$

2.5 The Charge Contribution

Before we begin our full analysis on the electromagnetic terms, we must first look at the contribution of the charge. This section can in general be seen as an extension of the treatment of the charge in [16] to a metric satisfying our criteria.

First, for any divergence-free quantity J, we have the definition (where we integrate with respect to the Euclidean metric on $\mathbb{R}^3$)

$$q(t) = \int_{\Sigma_t} -\sqrt{|g|} J^0. \tag{2.40}$$

Since J is divergence-free, we can drop the dependence on time.

Likewise,

$$\partial_\alpha \left(\sqrt{|g|} F^{0\alpha}\right) = \sqrt{|g|} J^0, \tag{2.41}$$

which follows from the antisymmetry of F. Since $F^{00} = 0$, we can write this as a spatial divergence on time slices with respect to the Euclidean metric. We can take the Hodge decomposition into

$$\sqrt{|g|} F^{0i} = \tilde{E}^i_{df} + \tilde{E}^i_{cf}, \tag{2.42}$$

the (Euclidean) divergence-free and curl-free parts respectively. We take a potential function Φ for the curl-free part, writing

$$\tilde{E}^i_{cf} = \partial^i \Phi,$$

secondary where again indices are raised according to the Euclidean metric. It follows that Φ satisfies the Laplace equation

$$\Delta\Phi = \sqrt{|g|}J^0$$

It follows from elliptic consideration that $\partial\Phi$ cannot decay faster than r^{-2} unless the space integral of the right hand side integrates to 0.

We define the charge 1-form as follows:

$$\overline{A} = \left(\int_0^r \frac{q}{4\pi} \frac{\overline{\chi}(s^* - t - 2)\partial_s(s^*)}{s^{*2}} \, ds \right) dt, \tag{2.43}$$

where s^* is defined to be analogous with r^*; i.e. $s^* = s + M\chi \ln(1+s)$. Additionally, $\overline{\chi}$ is a smooth increasing function satisfying

$$\overline{\chi}(y) = \begin{cases} 1 & y > 1, \\ 0 & y < 0, \end{cases} \tag{2.44}$$

We can now define $\overline{F} = d\overline{A}$, or

$$\overline{F}_{0i} = \omega^i \left(\frac{q}{4\pi} \frac{\overline{\chi}(r^* - t - 2)\partial_r(r^*)}{r^{*2}} \right). \tag{2.45}$$

It is easy to see that this has the same decay as $\tilde{E}^i_{cf}$, up to terms decaying like $Mr^{-3}\ln(r)$. We now look at the null decomposition. Straightforward calculation gives us

$$\frac{1}{2} F_{\underline{L}^* L^*} = \left(\frac{q}{4\pi} \frac{\overline{\chi}(r^* - t - 2)}{r^{*2}} \right), \tag{2.46a}$$

$$\alpha(F) = \underline{\alpha}(F) = \sigma(F) = 0. \tag{2.46b}$$

We can use this to establish component estimates on all Lie derivatives of $\overline{F}$. Fortunately our choice of

$\overline{F}$ makes this process relatively straightforward. We have in particular the estimates

$$|\alpha[\mathcal{L}_Z^I \overline{F}]| \lesssim |q|\tau_-\tau_+^{-3}, \tag{2.47a}$$

$$|\rho[\mathcal{L}_Z^I \overline{F}]| \lesssim |q|\tau_+^{-2}, \tag{2.47b}$$

$$|\sigma[\mathcal{L}_Z^I \overline{F}]| \lesssim |q|\tau_+^{-2}, \tag{2.47c}$$

$$|\underline{\alpha}[\mathcal{L}_Z^I \overline{F}]| \lesssim |q|\tau_+^{-2}. \tag{2.47d}$$

These follow from the commutator terms (2.5), using an analogous argument to Lemma (2.2).

We put off discussion of the associated charge until later, noting for now that, as in the Minkowski case, the worst decaying part of this occurs along the region $\tau_- \approx 1$.

2.6 A Model Morawetz Inequality

Here we prove a model Morawetz inequality, which will hopeful cast light on the reasoning for the modifications we make. Here we consider solutions to the equation

$$\partial_\alpha(g^{\alpha\gamma}\partial_\gamma\phi) = 0$$

in Minkowski space. This is precisely the geometric wave operator with respect to the metric g if the determinant of g is -1; otherwise, we can replace $g^{\alpha\beta}$ with $\sqrt{|g|}g^{\alpha\beta}$ and achieve similar estimates.

We take the null frame $\{L = \partial_t + \partial_r, \underline{L} = \partial_t - \partial_r, S_j\}$, where S_j are piecewise defined orthonormal fields tangent to spheres of fixed radius. Additionally, we have the optical weights in Minkowski space

$$\tau_+^2 = 1 + (t+r)^2 \qquad \tau_-^2 = 1 + (t-r)^2, \qquad \tau_0^2 = \tau_-^2/\tau_+^2.$$

These are consistent with our modified null decomposition and optical weights for $M = 0$.

We state our estimate as follows.

Theorem 2.7. *There exists a constant $\epsilon > 0$ such that for a compact function ϕ, and $H^{\alpha\beta}$ satisfying*

$$\tau_+|L(H_{\mathcal{L}\mathcal{L}})| + \tau_-|\underline{L}(H_{\mathcal{L}\mathcal{L}})| + |H_{\mathcal{L}\mathcal{L}}| \le \epsilon\tau_0^{-2}, \tag{2.48a}$$

$$\tau_+|L(H_{\mathcal{T}\mathcal{U}})| + \tau_-|\underline{L}(H_{\mathcal{T}\mathcal{U}})| + |H_{\mathcal{T}\mathcal{U}}| \le \epsilon\tau_0^{-1}, \tag{2.48b}$$

$$\tau_+|L(H_{\underline{L}\underline{L}})| + \tau_-|\underline{L}(H_{\underline{L}\underline{L}})| + |H_{\underline{L}\underline{L}}| \le \epsilon, \tag{2.48c}$$

where indices are raised and lowered according to the Minkowski metric, and the energy

$$\mathcal{E}(t) = \int_{\Sigma_t} \left(\frac{\tau_+^2}{4} \left(\frac{L(r\phi)}{r} \right)^2 + \frac{\tau_-^2}{4} \left(\frac{\underline{L}(r\phi)}{r} \right)^2 + \frac{\tau_+^2 + \tau_-^2}{4} \sum_j |S_j \phi|^2 \right) dx \tag{2.49}$$

we have the estimate

$$\mathcal{E}(T) \lesssim \mathcal{E}(0) + \epsilon \int_0^T \frac{\mathcal{E}(t)}{1+t} + \int_0^T \int_{\Sigma_t} \left| \frac{K_0(r\phi)}{r} \partial_\alpha (g^{\alpha\gamma} \partial_\gamma \phi) \right| \tag{2.50}$$

This is a slight improvement over a similar argument in [8], in that we require less decay on certain components of the metric. However, as we will see, the bound on H_{LL} is sharp, and anything worse would result in the possibility of exponentially growing energy. We also have that the bound on H_{LL} and derivatives is the best we can hope for, even after subtracting off the mass part of the metric (which introduces substantial additional complications in itself). This follows analysis in [13].

It follows from Lemma 9.7 that $\mathcal{E}(t)$ is equivalent to the term

$$\int_{\Sigma_t} \tau_+^2 |L\phi|^2 + \tau_-^2 |\underline{L}\phi|^2 + \sum_j \tau_+^2 |S_j \phi|^2 + \tau_+^2 \left| \frac{\phi}{r} \right|^2 dx, \tag{2.51}$$

where we first bound the far right term by the energy and use that to bound the other terms. In general we use these interchangeably.

Proof. We first take the vector field

$$K_0 = \frac{1}{2}(\tau_+^2)(\partial_t + \partial_r) + \frac{1}{2}(\tau_-^2)(\partial_t - \partial_r) = (1 + t^2 + r^2)\partial_t + 2tr\partial_r. \tag{2.52}$$

We note that K_0 is conformal Killing (but not Killing) with respect to the Minkowski metric.

Our main tool is the divergence theorem applied to the quantity

$$P^\alpha = -\left(\frac{K_0(r\phi)}{r} g^{\alpha\gamma} \partial_\gamma \phi - \frac{1}{2} K_0^\alpha g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi + \frac{1}{2}(L^\alpha + \underline{L}^\alpha)\phi^2 \right). \tag{2.53}$$

Integrating along time slices, with $\alpha = 0$, gives us the quantity

$$E(t) = \int_{\Sigma_t} P^0 = \int_{\Sigma_t} \left(\frac{K_0(r\phi)}{r} \partial_t \phi + \frac{\tau_+^2 + \tau_-^2}{4} (-L\phi \underline{L}\phi + |S_j\phi|^2) - \phi^2 \right) \tag{2.54}$$
$$+ \left(-\frac{K_0(r\phi)}{r} (H^{L\gamma} \partial_\gamma \phi + H^{\underline{L}\gamma} \partial_\gamma \phi) + \frac{1}{4} (\tau_+^2 + \tau_-^2) H^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi \right),$$

We therefore have

$$E(T) - E(0) = \int_0^T \int_{\Sigma_t} \partial_\alpha P^\alpha \, dx \, dt. \tag{2.55}$$

We must therefore show that the quantities $E(0)$, $E(T)$ are equivalent to the energies $\mathcal{E}(0)$, $\mathcal{E}(T)$, and that the integral on the right hand side is bounded by the right hand side of (2.50).

We can write

$$\partial_t \phi = \frac{1}{2} \left(\frac{L(r\phi)}{r} + \frac{\underline{L}(r\phi)}{r} \right), \qquad -L\phi \underline{L}\phi = -\left(\frac{L(r\phi)}{r} \right) \left(\frac{\underline{L}(r\phi)}{r} \right) - \frac{2\partial_r (r\phi)}{r} \frac{\phi}{r} + \left(\frac{\phi}{r} \right)^2$$

which lets us rewrite the Minkowski part of $E(t)$ (the first line of (2.54)) as

$$\int_{\Sigma_t} \left(\frac{\tau_+^2}{4} \left(\frac{L(r\phi)}{r} \right) + \frac{\tau_-^2}{4} \left(\frac{\underline{L}(r\phi)}{r} \right) + \frac{\tau_+^2 + \tau_-^2}{4} \left(\sum_j |S_j\phi|^2 - \frac{2\partial_r (r\phi)}{r} \frac{\phi}{r} + \left(\frac{\phi}{r} \right)^2 \right) - r^2 \left(\frac{\phi}{r} \right)^2 \right). \tag{2.56}$$

We have the identity

$$\partial_i \left[\left(\frac{\tau_+^2 + \tau_-^2}{4} \frac{\omega_i \phi^2}{r} \right) \right] = \frac{\tau_+^2 + \tau_-^2}{4} \frac{\phi^2}{r^2} + \frac{\tau_+^2 + \tau_-^2}{4} \frac{2\phi \partial_r \phi}{r} + \phi^2 + \left(\frac{\tau_+^2 + \tau_-^2}{4} \frac{\phi^2}{r} \right). \tag{2.57}$$

By the divergence theorem we have that the integral of this over space is 0 as long as ϕ has sufficient decay at spatial infinity. We can therefore add this integral to the Minkowski part of $E(t)$ without consequence. Noting cancellations, the Minkowski part of $E(t)$ is equal to

$$\int_{\Sigma_t} \left(\frac{\tau_+^2}{4} \left(\frac{L(r\phi)}{r} \right) + \frac{\tau_-^2}{4} \left(\frac{\underline{L}(r\phi)}{r} \right) + \sum_j \frac{\tau_+^2 + \tau_-^2}{4} |S_j\phi|^2 \right), \tag{2.58}$$

which is precisely our energy $\mathcal{E}(t)$.

Now we consider the error terms (the second line of (2.54)). We recall the estimates

$$|H_{\mathcal{L}\mathcal{L}}| \lesssim \epsilon\tau_0^{-2}, \tag{2.59}$$

$$|H_{\mathcal{T}\mathcal{U}}| \lesssim \epsilon\tau_0^{-1}, \tag{2.60}$$

$$|H_{\underline{L}\underline{L}}|, |H_{\underline{L}S_i}| \lesssim \epsilon. \tag{2.61}$$

It suffices to show that the error terms can be bounded uniformly by $\frac{1}{2}\mathcal{E}(t)$. We show this for the terms containing $H^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi$. Other terms follow similarly.

We have the inequality

$$|H^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi| \lesssim \epsilon\tau_0^2|\underline{L}\phi|^2 + \epsilon\tau_0|\underline{L}\phi||\overline{\partial}\phi| + \epsilon|\overline{\partial}\phi|^2 \lesssim \epsilon\tau_0^2|\underline{L}\phi|^2 + \epsilon|\overline{\partial}\phi|^2. \tag{2.62}$$

We recall the notation

$$|\overline{\partial}\phi|^2 = |L\phi|^2 + \sum_j |S_j\phi|^2$$

.

We then have the estimate

$$\begin{aligned}
\int_{\Sigma_t} \frac{1}{4}(\tau_-^2 + \tau_+^2)|H^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi| &\lesssim \int_{\Sigma_t} \epsilon\tau_0^2\tau_+^{2s}|\underline{L}\phi|^2 + \epsilon\tau_+^{2s}|\overline{\partial}\phi|^2, \\
&\lesssim \epsilon\int_{\Sigma_t} \tau_+^{2s}|\overline{\partial}\phi|^2 + \tau_-^{2s}|\underline{L}\phi|^2, \\
&\lesssim \epsilon\mathcal{E}(t).
\end{aligned}$$

This can be bounded by an arbitrarily small constant times $\mathcal{E}(t)$ given sufficiently small ϵ.

We note that the requirement on $H_{\mathcal{L}\mathcal{L}}$ comes from the term like $\tau_+^2 H^{\gamma\delta}\partial_\gamma\phi\partial\delta\phi$, as we need the terms like $\tau_+^2 H_{LL}|\underline{L}\phi|^2$ to behave like $\epsilon\tau_-^2|\underline{L}\phi|^2$ in order to be contained in our initial energy. This turns out to be the limiting term that derives many of the modifications we use.

We now look at the spacetime integral of the divergence $\partial_\alpha P^\alpha$.

$$\partial_\alpha P^\alpha = \left[\partial_\alpha\left(-\left(\frac{K_0(r\phi)}{r}g^{\alpha\gamma}\partial_\gamma\phi - \frac{1}{2}K_0^\alpha g^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi + \frac{1}{2}(L^\alpha + \underline{L}^\alpha)\phi^2\right)\right) + \frac{K_0(r\phi)}{r}\partial_\alpha(g^{\alpha\gamma}\partial_\gamma\phi)\right] - \tag{2.63}$$

$$-\left[\frac{K_0(r\phi)}{r}\partial_\alpha(g^{\alpha\gamma}\partial_\gamma\phi)\right] \tag{2.64}$$

where $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$, $m^{\alpha\beta}$ is the inverse Minkowski metric.

The second term on the right shows up on the right hand side of (2.50). We now focus on the first term on the right.

We first have the estimate

$$\partial_\alpha \left(\frac{K_0(r\phi)}{r} \right) = \partial_\alpha(2t\phi) + K_0^\beta \partial_\beta \partial_\alpha \phi + 2t\partial_\alpha \phi. \tag{2.65}$$

This follows from the identity

$$\partial_\alpha K_0^\beta = 2t\delta_\alpha^\beta.$$

Additionally, we have

$$-\frac{1}{2}\partial_\alpha \left(K_0^\alpha g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi \right) = -4tg^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi - \frac{1}{2}K_0(g^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi - K_0^\alpha g^{\gamma\delta}\partial_\gamma\phi\partial_\delta\partial_\alpha\phi. \tag{2.66}$$

Next, we have

$$\partial_\alpha \left(\frac{1}{2}(L^\alpha + \underline{L}^\alpha)\phi^2 \right) = 2\phi\partial_t\phi. \tag{2.67}$$

Therefore, we have

$$(2.63) = -\frac{1}{2}K_0(g^{\gamma\delta})\partial_\gamma\phi\partial_\delta\phi + 2g^{\alpha\gamma}\phi\partial_\gamma\phi\partial_\alpha(t) - 2\phi\partial_t\phi. \tag{2.68}$$

Our goal is to bound the spacetime integrals on these quantities in magnitude. We consider the region $r > \frac{t}{2} + 1$, as the far interior is easier, as we do not need to distinguish weights or derivatives. We consider the first term. We can decompose g in our null frame and take the vectors outside the derivative, as in general the error terms satisfy the same or nicer estimates. By our metric decomposition we have

$$|K_0(g^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi| \lesssim \epsilon \left(|K_0(H_{LL})||\underline{L}\phi|^2 + |K_0(H_{\mathcal{TU}})||\partial\phi||\overline{\partial}\phi| + |K_0(H_{\mathcal{UU}})|||L\phi|^2 \right),$$

where

$$|\overline{\partial}\phi|^2 = |L\phi|^2 + |\not{\partial}\phi|^2.$$

Using our bounds on H, we can bound this by

$$\epsilon \left(\tau_-^2\tau_+^{-1}|\partial\phi|^2 + \tau_-|\partial\phi||\overline{\partial}\phi| + \tau_+|\overline{\partial}\phi|^2 \right).$$

We can ignore the middle term, as we can bound it using the two other terms. The integral of these in space is easily bounded in magnitude by

$$\epsilon\tau_+^{-1}\mathcal{E}(t),$$

so the spacetime integral is on the right hand side of (2.50). Now we look at

$$2g^{\alpha\gamma}\phi\partial_\gamma\phi\partial_\alpha(t) - 2\phi\partial_t\phi = H^{\alpha\gamma}\phi\partial_\gamma\phi\partial_\alpha t.$$

It suffices to bound the spacetime norm of

$$H^{L\gamma}\phi\partial_\gamma\phi,$$

as the other term has nicer decay in H. We need to bound

$$|H_{\underline{LL}}||\phi||L\phi| + |H_{\mathcal{TU}}||\phi||\partial\phi|.$$

We write

$$\phi = \tau_+\left|\frac{\phi}{\tau_+}\right|,$$

and take the bounds

$$|H_{\underline{LL}}||\phi||L\phi| \lesssim \epsilon\tau_+\left(\left|\frac{\phi}{\tau_+}\right|^2 + |\overline{\partial}\phi|^2\right) \tag{2.69a}$$

$$|H_{\mathcal{TU}}||\phi||\partial\phi| \lesssim \epsilon\tau_+\left|\frac{\phi}{\tau_+}\right|^2 + \tau_-^2\tau_+^{-1}|\partial\phi|^2 \tag{2.69b}$$

These are similarly bounded in magnitude by the right hand side of (2.50). □

This estimate in itself is not particularly useful to us for two reasons. First, this estimate combined with Gronwall's inequality gives slowly growing energy, which is not sufficient for our needs. In particular, certain estimates coming from the commutator require sharp bounds in time. In order to achieve bounded energy, we would need H_{LL} to decay like $\tau_-^2\tau_+^{-2-\epsilon}$ for some $\epsilon > 0$, along with an extra power of $\tau_+^{-\epsilon}$ on other components, which is not realistic.

Second, due to the mass, the best decay we can expect on the light cone for even good components of H is τ_+^{-1}.

We note that the sharp decay necessary for H_{LL} comes from the fact that in the geometric estimate in

Minkowski space, one term which shows up as an error term behaves like

$$(\mathcal{L}_{K_0}g)^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi.$$

In general, we can integrate by parts or use a conformal transformation of the metric in order to replace g with its error terms H. However, since

$\tau_+^2 H_{LL} \lesssim \epsilon\tau_-^2$.

One way of requiring less decay would be to use the fractional Morawetz estimate used by Lindblad and Sterbenz in [16]. Given this estimate, we would only need decay like $\tau_0^{2s}\tau_+^{-\epsilon}$ for some $\epsilon > 0$, and analogous terms for other metric components, in order to be able to apply Gronwall's Lemma to get bounded energy. However, we still have decay like τ_+^{-1} for the best components of H, due to the mass, so for the fractional Morawetz estimate with $s \geq 1/2$ this still does not give us bounded energy.

One natural way to accomplish this is by modifying our null frame somehow. We note that the fact that our energy estimate comes from the fact that K_0 is conformal Killing in Minkowski; i.e.

$$\mathcal{L}_{K_0}m - 4tm = 0.$$

When we introduce a metric, we of course get small error terms, so it suffices to bound these error terms. For the standard null frame in Minkowski, if we take the Morawetz estimate from a geometric perspective, we get error terms which behave like

$$|(\mathcal{L}_{K_0}g)_{LL} - 4tg_{LL}||\underline{L}\phi|^2 \lesssim \epsilon|\underline{L}\phi|^2,$$

which is insufficient for our needs. This bad component of this in particular comes from the massive part of the metric.

Note in particular that in order to get bounded energy, we would need decay like $\tau_+^{-1-\epsilon}$ along the light cone in this term. Even with the fractional Morawetz estimate, we get decay like τ_+^{2s-2}, which leads to growing energy even for the boundary case $s = 1/2$. Therefore, we must adjust our estimate by taking into account the massive part of the metric.

We could of course select our vector L such that $g_{LL} = 0$, which would establish our estimate at the cost of having to establish estimates on many consequent terms (see for instance [4]). A computationally simpler approach would be to select a vector which is null, or sufficiently close, with respect to the first-order metric

$\tilde{m}$. The estimates in [13] in particular give us the decay we need, as long as we select a modified vector L^* such that the contribution of the quantity $m + (M\chi/(1+r))\delta$ to the terms coming from the deformation tensor is sufficiently small. We now consider our modified fields L^*, along with the decomposition $g_{\alpha\beta} = \tilde{m}_{\alpha\beta} + h^1_{\alpha\beta}$. We first have the estimate

$$\tilde{m}_{L^*L^*} \lesssim M\tau_-^{1+\iota}\tau_+^{-2}. \tag{2.70}$$

This is a consequence of (2.29), noting that close to the light cone, we have the estimates

$$m\left(L^*, L^*\right) = -1 + \left(1 - \frac{M\chi}{1+r}\right)^2 + M \cdot O(\tau_+^{-2}) \qquad \frac{M\chi}{1+r}\delta(L^*, L^*) = 2\frac{M\chi}{1+r} + M \cdot O(\tau_+^{-2}). \tag{2.71}$$

Therefore, the sum is of order $M \cdot O(\tau_+^{-2})$. We have worse decay in the far interior, of order $M \cdot O(\tau_+^{-1}\ln(\tau_+))$; however, we can compensate for this by multiplying by a power of τ_0, which is bounded below in this region.

$$\begin{aligned}
(\mathcal{L}_{\overline{K_0^s}} g)_{L^*L^*} &= \mathcal{L}_{\overline{K_0^s}}(g_{L^*L^*}) - 2g([\overline{K_0^s}, L^*], L^*), \\
&\lesssim \epsilon\tau_-^{\gamma'}\tau_+^{2s-2-\gamma'}, \\
&\lesssim \epsilon\tau_0^{2s}\tau_+^{-\epsilon}.
\end{aligned}$$

This follows from the fact that these estimates are true for both $\tilde{m}$, since we have chosen L^* precisely so that $\tilde{m}_{L^*L^*}$ satisfies certain decay estimates, and h^1, by estimates in [13], and is precisely the decay we desire in the fractional Morawetz estimate for these components. Therefore, the modified frame is generally suitable for these kinds of estimates in relativistic metrics.

We briefly sketch such an improved estimate as follows. This is mostly a simplified version of the future estimate (4.1), so we can streamline our proof and leave proofs of certain minor details for the future estimate.

Lemma 2.8. *Given the metric $\tilde{m}$, and the energy*

$$E_0[\phi](T) = \int_{\Sigma_t} \left(\tau_+^2\left(\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + |\not{D}\phi|^2\right) + \tau_-^2\left|\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|^2\right) dx.$$

as defined in (4.3) *for $s = 1$, given a function ϕ, we have the estimate*

$$E_0[\phi](T) - E_0[\phi](0) \lesssim M\int_0^T \frac{E_0[\phi](t)\ln(1+t)}{1+t} + \int_0^T\int_{\Sigma_t}\left|\frac{K_0^*(r^*\phi)}{r^*}\Box_{\tilde{m}}\phi\right|. \tag{2.72}$$

Alternatively, we can write

$$E_0[\phi](T) - E_0[\phi](0) \lesssim M \int_0^T \frac{E_0[\phi](t)}{1+t} + M \int_0^T \int_{\Sigma_t} \tau_-^{-2}\tau_+^2 |L^*\phi|^2 + \int_0^T \int_{\Sigma_t} \left| \frac{K_0^*(r^*\phi)}{r^*} \Box_{\tilde{m}}\phi \right|. \tag{2.73}$$

Neither of these are useful in themselves. For (2.72) an application of Gronwall's Lemma gives us $(1+t)^{M\ln(1+t)}$ growth of the energy. However, any additional polynomial decay in the error terms (i.e. replacing the factor $(\ln(1+t)/(1+t))$ with $(\ln(1+t)/(1+t)^{1+\epsilon})$ gives us bounded energy.

In (2.73), we have more slowly growing energy coming from the first term on the right (like $(1+t)^{CM}$ for some constant C); however, we do not easily have bounds on the weighted spacetime integral of $|L^*\phi|^2$. This integral, however, can be bounded by introducing a weight which is bounded in the interior and growing like $1+(r-t)^\gamma$ in the exterior. This is analogous to the spacetime integral method used in [16], as well as in [15].

We will show that this additional decay follows from the use of the fractional Morawetz estimate.

Proof. Here all geometric expressions, including ∇, will be with respect to the metric $\tilde{m}$.

We take the geometric divergence estimate on the momentum density tensor

$$-\frac{1}{r^{*2}} Q[r^*\phi]_{\alpha\beta} K_0^{*\beta}, \tag{2.74}$$

where

$$K_0^* = (1+(t+r^*)^2)L^* + (1+(t-r^*)^2)\underline{L}^*. \tag{2.75}$$

The time-slice energy is equal to

$$-\frac{1}{r^{*2}} \sqrt{|\tilde{m}|}\,\tilde{m}^{0\gamma} Q[r^*\phi]_{\gamma\beta} K_0^{*\beta},$$

which is equivalent to the quantity $E_0[\phi](T)$. This equivalency follows from expanding everything out, combined with Lemma 9.8. Therefore, we have the estimate

$$E_0[\phi](T) - E_0[\phi](0) \lesssim \int_0^T \int_{\Sigma_t} \left| \nabla^\alpha \left(\frac{1}{r^{*2}} Q[r^*\phi]_{\alpha\beta} K_0^{*\beta} \right) \sqrt{|\tilde{m}|} \right| dx\, dt. \tag{2.76}$$

In general we can leave out the volume element as it is close to 1.

We take the divergence

$$\nabla^\alpha \left(\frac{1}{r^{*2}} Q[r^*\phi]_{\alpha\beta} K_0^{*\beta} \right) = -\frac{2\nabla^\alpha(r^*)}{r^{*3}} Q[r^*\phi]_{\alpha\beta} K_0^{*\beta} + \frac{1}{r^*} \Box_{\tilde{m}}(r^*\phi) \frac{K_0^*(r^*\phi)}{r^*} + \frac{1}{r^{*2}} (\nabla^\alpha K_0^{*\beta}) Q[r^*\phi]_{\alpha\beta}. \tag{2.77}$$

We use equation (4.1) to rewrite

$$\frac{1}{r^*}\Box_{\tilde{m}}(r^*\phi)\frac{K_0^*(r^*\phi)}{r^*} = \left(\Box_{\tilde{m}}\phi + r^*\phi\Box_{\tilde{m}}\frac{1}{r^*} - 2\nabla^\alpha\left(\frac{1}{r^*}\right)\nabla_\alpha(r^*\phi)\right)\left(\frac{K_0^*(r^*\phi)}{r^*}\right).$$

We note that the term with $\Box_{\tilde{m}}\phi$ appears on the right hand side of (2.72). The term containing $\Box_{\tilde{m}}\left(\frac{1}{r^*}\right)$ can be ignored, as we can establish nice bounds for this term without trouble. Finally, the term with $\nabla^\alpha\left(\frac{1}{r^*}\right)$ cancels out

$$-\frac{2\nabla^\alpha(r^*)}{r^{*3}}\partial_\alpha(r^*\phi)\partial_\beta(r^*\phi)K_0^{*\beta}$$

in the first term on the right hand side of (2.77). The only terms we haven't dealt with are

$$-\frac{2\nabla^\alpha(r^*)}{r^{*3}}\tilde{m}_{\alpha\beta}\tilde{m}^{\gamma\delta}\partial_\gamma(r^*\phi)\partial_\delta(r^*\phi)K_0^{*\beta} + \frac{1}{r^{*2}}(\nabla^\alpha K_0^{*\beta})Q[r^*\phi]_{\alpha\beta}.$$

For the first term, we can immediately establish

$$-\frac{2\nabla^\alpha(r^*)}{r^{*3}}\tilde{m}_{\alpha\beta}K_0^{*\beta} = \frac{2}{r^{*3}}K_0^*(r^*) = -\frac{4t}{r^{*2}}.$$

It suffices to bound

$$\frac{1}{r^{*2}}\left(-4t\tilde{m}^{\gamma\delta}\partial_\gamma(r^*\phi)\partial_\delta(r^*\phi) + (\nabla^\alpha K_0^{*\beta})Q[r^*\phi]_{\alpha\beta}\right).$$

We establish estimates on the latter term. We have in particular the commutator identities

$$[K_0^*, L^*] = -2\underline{u}^*L^*, \tag{2.78a}$$

$$[K_0^*, \underline{L}^*] = -2u^*\underline{L}^*, \tag{2.78b}$$

$$[K_0^*, S_j^*] = -2tS_j^*. \tag{2.78c}$$

It follows that for $X, Y \in \{L^*, \underline{L}^*, S_j^*\}$,

$$(\mathcal{L}_{K_0^*}\tilde{m})_{XY} = K_0^*(\tilde{m}_{XY}) + 4t\tilde{m}_{XY}. \tag{2.79}$$

Therefore, recalling the identity (2.23), we have the decomposition

$$\frac{1}{r^{*2}}\left(-4t\tilde{m}^{\gamma\delta}\partial_\gamma(r^*\phi)\partial_\delta(r^*\phi) + (\nabla^\alpha K_0^{*\beta})Q[r^*\phi]_{\alpha\beta}\right) = K_0^*(\tilde{m}_{XY})Q[\phi]^{XY}. \tag{2.80}$$

When $X, Y \in \{L^*, \underline{L}^*\}$ (not necessarily equal), we have that

$$|K_0^*(\tilde{m}_{XY})| \lesssim M\tau_+^{-1}\tau_- \ln(1+\tau_-),$$

and when $X = Y = S_j^*$, we have that

$$|K_0^*(\tilde{m}_{XY})| \lesssim M \ln(1+\tau_+),$$

with all other terms equal to 0. These follow from (2.29), with the note that in the $L^*, \underline{L}^*$ components we have our worst decay in the far interior, which comes from the fact that $\partial_t(r^*)$ decays like $\tau_+^{-1}\ln(t)$ and has no other term to cancel it out.

Expanding the components of Q gives us the bound

$$|K_0^*(\tilde{m}_{XY})Q[\phi]^{XY}| \lesssim M\left(\tau_+^{-1}\ln(\tau_+)\tau_-^2|\partial\phi|^2 + \tau_+\ln(\tau_+)|\overline{\partial}\phi|^2\right),$$

which establishes (2.72), or

$$|K_0^*(\tilde{m}_{XY})Q[\phi]^{XY}| \lesssim M\left(\tau_+^{-1}\tau_-^2|\partial\phi|^2 + \tau_+|\overline{\partial}\phi|^2 + \tau_+^2\tau_-^{-2}\ln(1+\tau_+)^2|L^*\phi|^2\right),$$

which establishes (2.73).

This follows from the fact that

$$|K_0^*(\tilde{m}_{L^*L^*})Q[\phi]^{L^*L^*}| \lesssim M\tau_+^{-1}\tau_-^{1+\iota}|\partial\phi|^2,$$

along with

$$|K_0^*(\tilde{m}_{S_j^*S_j^*})Q[\phi]^{S_j^*S_j^*}| \lesssim M\ln(\tau_+)\left(\sum_i |S_i^*\phi|^2 + |L^*\phi||\underline{L}^*\phi| + \text{ error terms.}\right).$$

The latter formula follows from writing $Q^{S_j^*S_j^*} = \tilde{m}^{S_j^*\alpha}Q_{\alpha\beta}\tilde{m}^{\beta S_j^*}$ and expanding the terms containing the inverse metric $\tilde{m}$ in our null frame, which gives us $|Q^{S_j^*S_j^*}| \lesssim |Q_{S_j^*S_j^*}|$.

We can additionally write

$$M\ln(\tau_+)\left(|L^*\phi||\underline{L}^*\phi|\right) \lesssim M\ln(\tau_+)^2\tau_+\tau_-^{-1}|L^*\phi|^2 + \tau_+^{-1}\tau_-^1|\underline{L}^*\phi|^2$$

(2.72) and (2.73) both follow. $\square$

In order to achieve bounded energy we must make two changes: first, we must account for the bad decay of terms like $K_0^*(\tilde{m}(L^*, L^*))|\underline{L}^*\phi|^2$, and second, we must deal with the bad angular terms. In each case, we get significant improvement by using the fractional Morawetz estimate, as we have the estimates

$$|\overline{K_0^s}(\tilde{m}_{L^*L^*})| \lesssim M\tau_+^{2s-3}\tau_-^{1+\iota},$$

as well as

$$|\overline{K_0^s}(\tilde{m}_{S_j^*S_j^*})| \lesssim M\ln(\tau_+)\tau_+^{2s-2},$$

both of which lead to greater decay and therefore boundedness of the energy in the range $s < 1$. (see for instance the inequality (4.11e) in the latter case).

We have one last estimate, which shows the use of the energy (2.73).

Lemma 2.9. *Given the metric $\tilde{m}$, and the energy*

$$E_w[\phi](T) = \int_{\Sigma_T} \left(\tau_+^2\left(\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + |\not{D}\phi|^2\right) + \tau_-^2\left|\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|^2\right) w\,dx,$$

along with the interior spacetime energy

$$S_w[\phi](T) = \int_0^T \int_{\Sigma_t} \left(\tau_+^2|L\phi|^2 + \tau_-^2\sum_j |S_j\phi|^2\right) w'\,dx\,dt,$$

where

$$w = \begin{cases} 1 + (1 + (t - r^*))^{-\iota} & r^* < t, \\ 1 + (1 + (t + r^*))^{\delta} & r^* > t, \end{cases} \qquad w' = \begin{cases} (1 + (t - r^*))^{-1-\iota} & r^* < t, \\ 1 + (1 + (t + r^*))^{\delta-1} & r^* > t, \end{cases}$$

for some constants $\iota, \delta > 0$, we have the estimate

$$E_w[\phi](T) + S_w[\phi](T) \lesssim_{\iota,\delta} E_w[\phi](0) + M\int_0^T \frac{E_w[\phi](t)}{1+t} + \int_0^T \int_{\Sigma_t} \left|\frac{K_0^*(r^*\phi)}{r^*}\Box_{\tilde{m}}\phi w\right|. \tag{2.81}$$

Proof. This almost exactly follows the proof of the estimate (2.73), where we instead take the divergence

$$-\nabla^\alpha\left(\frac{1}{r^{*2}}Q[r^*\phi]_{\alpha\beta}K_0^{*\beta}w\right) = \frac{2\nabla^\alpha(r^*)}{r^{*3}}Q[r^*\phi]_{\alpha\beta}K_0^{*\beta}w - \frac{1}{r^*}\Box_{\tilde{m}}(r^*\phi)\frac{K_0^*(r^*\phi)}{r^*}w - \tag{2.82}$$
$$- \frac{1}{r^{*2}}(\nabla^\alpha K_0^{*\beta})Q[r^*\phi]_{\alpha\beta}w - \frac{1}{r^{*2}}Q[r^*\phi]_{\alpha\beta}K_0^{*\beta}\nabla^\alpha w.$$

We can bound the first three terms on the right identically to (2.73). For the fourth terms, we need to look at estimates on w. We have that

$$\nabla^\alpha(w) = \tilde{m}^{\alpha\underline{L}^*}\underline{L}^*(w) = \left(-\frac{1}{2}L^{*\alpha} + M\cdot O(\tau_-\ln(\tau_-)\tau_+^{-2})L^{*\alpha} + M\cdot O(\tau_-\ln(\tau_-)\tau_+^{-2})\underline{L}^{*\alpha}\right)\underline{L}^*(w),$$

since all other derivatives of w are equal to 0. Taking the estimate $\underline{L}^*(w)\approx -w'$, in the sense that $C^{-1}w'\leq -\underline{L}^*(w)\leq Cw'$ for some positive C, we see that the first term contributes

$$\int_0^T\int_{\Sigma_t} -\frac{1}{2}\frac{1}{r^{*2}}Q[r^*\phi](L^*,K_0^*)w'\,dx\,dt \tag{2.83}$$

to the integral of the divergence. This is a signed quantity equivalent to $-S_w[\phi](T)$ modulo error terms which can be bounded without issue. We can therefore add it to the left hand side without issue.

Given this, to close the argument, we need to show that the extra spacetime integral on the right hand side of (2.73) can be subtracted off from $S_w[\phi](T)$ without issue, for sufficiently small M. It suffices to show that there exists an M small enough that

$$S_w[\phi](T) - CM\int_0^T\int_{\Sigma_t}\tau_-^{-2}\tau_+^2|L^*\phi|^2 \geq \frac{1}{2}S_w[\phi](T),$$

where C is the constant implicit in the $\lesssim$ in (2.73) times the one implicit in the comparison of $-S_w[\phi](T)$ to the spacetime integral (2.83).. This follows from expanding $S_w[\phi](T)$ out and using the relation $w\tau_-^{-2}\leq w'$. $\square$

3 The L^2 Estimate: F

We now establish an energy estimate on the electromagnetic field F. Our basic approach here follows the fractional Morawetz estimate used in [16], with the substitution of the modified vector fields, for the reasons mentioned previously, and with additional calculations taken in order to bound the error terms.

Our method is fundamentally based on the conformal Morawetz inequality, suitably adapted to the decay of the initial data and to our metric. We first construct the warped vector field

$$\overline{K_0^s} = \frac{1}{2}\left((1+\underline{u}^{*2s})L^* + (1+|u^*|^{2s})\underline{L}^*\right) \tag{3.1}$$

for $1/2 < s < \gamma' < 1$, which can be seen as an interpolation between the fields $Z = \frac{1}{2}\left(\underline{u}^* L^* + u^* \underline{L}^*\right)$ (inside the light cone) and $K_0 = \frac{1}{2}\left(\underline{u}^{*2} L^* + u^{*2} \underline{L}^*\right)$, which correspond to conformal Killing fields in Minkowski space. We also add the field ∂_{t^*} in order to ensure that we have a timelike field close to the light cone given certain smallness conditions on the metric.

As usual, we contract this with our energy-momentum tensor on F and take the divergence theorem, first on time slabs to get a time-slice energy, then on regions of time slabs exterior to some forward light cone $u^* = c$, which will give us an additional term. We note that this additional term, coming from the integral along the light cone, is not necessarily positive definite; however, we can bound it below in a meaningful way. Additionally, we introduce a weight which will allow us to take certain spacetime estimates for terms coming from the scalar field which are sharp in radial decay.

We can treat certain terms coming from the metric as error terms. Importantly, in contrast with [15] and other analysis which uses the null condition, from a geometric standpoint we cannot treat the part of the metric coming from the mass (called h_0 in [15]) as purely an error term. This is because we require the nice component, ${}^{(\overline{K_0^s})}\pi_{L^* L^*}$, to decay better than t^{-1}. This is in particular not possible with the standard null frame and Lorentz fields in Minkowski, as for these terms we have fixed decay scaling like Mt^{-1}.

The primary estimate taken in this section will be a "naive" estimate which does not apply directly to our field F. This is because the energy on time slices is bounded only if the charge is 0. However, the charge decomposition, combined with certain elliptic estimates, will allow us to extend this estimate to the case with nonzero charge, at the cost of some extra terms.

Before we arrive at the statement for the basic estimate, we mention (and recall) some notational tools. First, we take the optical weight

$$\tau_0 = \tau_-/\tau_+. \tag{3.2}$$

Additionally, we define the weights

$$w = \tau_-^{2\delta}\overline{\chi}(r^* - t) + (1 - \overline{\chi}(r^* - t)), \tag{3.3a}$$

$$\tilde{w} = (1 + (2 - u^*)^{2\delta})\overline{\chi}(-u^*) + (1 + (2 + u^*)^{-2\iota})(1 - \overline{\chi}(u^*)) + \tag{3.3b}$$
$$+ (1 + \underline{u}^*)^{-2\iota}\left((2 - u^*)^{2\delta + 2\iota}\overline{\chi}(-u^*) + 1 - \overline{\chi}(-u^*)\right),$$

$$w_\iota = \tau_-^{2\delta}\overline{\chi}(r^* - t) + \tau_-^{2\iota}(1 - \overline{\chi}(r^* - t)), \tag{3.3c}$$

$$w' = \tau_+^{2\delta - 1}\overline{\chi}(r - t) + \tau_-^{-1-2\iota}(1 - \overline{\chi}(r^* - t)). \tag{3.3d}$$

Given s, we assume δ such that $\delta + s < 3/2$, and $0 < 2\iota < s - 1/2$. Here $\overline{\chi}$ is the same as in equation (2.44).

We briefly discuss the meaning of these four weights, which can be found in a similar form in [16]. Here, w is our basic exterior weight. We can think of it as auxiliary to the "peeling" weights τ_+^{2s} and τ_-^{2s}. $\tilde{w}$ is a weight which is equivalent to w, which behaves more nicely with respect to derivatives at the cost of increased complexity in the following sense:

$$\tilde{w} \approx w, \tag{3.4a}$$

$$-\frac{1}{2}\underline{L}^*(\tilde{w}) \approx w', \tag{3.4b}$$

$$-\frac{1}{2}L^*(\tilde{w}) \approx \tau_0^{1+2\iota}w'. \tag{3.4c}$$

These relations are straightforward to show and proofs can be easily adapted from analogous proofs in [16].

w' is a way to more easily express derivatives of $\tilde{w}$, and w_ι will be of use when we take L^2 estimates on the commutators, due to the relation

$$\tau_- w' w_\iota \approx w^2.$$

For the most part we will be able to use the relation

$$w_\iota \lesssim \tau_-^{2\iota} w,$$

except when dealing with certain terms coming from the charge.

Finally, we recall the modified null decomposition on two-forms $G_{\alpha\beta}$ as follows:

$$\alpha_j[G] = G_{L^* S_j^*}, \qquad \underline{\alpha}_j[G] = G_{\underline{L}^* S_j^*}, \tag{3.5a}$$

$$\rho[G] = \frac{1}{2} G_{\underline{L}^* L^*}, \qquad \sigma[G] = G_{S_1^* S_2^*}. \tag{3.5b}$$

As a notational convenience, we use the shorthand

$$|\not{G}|^2 = |\alpha[G]|^2 + |\rho[G]|^2 + |\sigma[G]|^2, \tag{3.6}$$

$$|G|^2 = |\alpha[G]|^2 + |\rho[G]|^2 + |\sigma[G]|^2 + |\underline{\alpha}[G]|^2, \tag{3.7}$$

where $|\alpha[G]|^2 = |\alpha_1[G]|^2 + |\alpha_2[G]|^2$, and $|\underline{\alpha}[G]|$ is defined similarly..

In the case where there is no ambiguity we drop the explicit dependence on G. Likewise, we can define the current on G,

$$J[G]_\alpha = \nabla^\beta G_{\alpha\beta}, \tag{3.8}$$

and finally, the spacetime weighted current L^2 norm

$$\|J\|_{L^2[w]} = \left\| \tau_+^s \tau_0^{-1/2-\iota} \tau_-^{1/2} J_{L^*} w_\iota^{1/2} \right\|_2 + \left\| \tau_+^s \tau_-^{1/2} |J_{S^*}| w_\iota^{1/2} \right\|_2 + \left\| \tau_0^{s-1/2-\iota} \tau_-^{s+1/2} |J_{\underline{L}^*}| w_\iota^{1/2} \right\|_2. \tag{3.9}$$

Theorem 3.1. *For any two-form $F_{\alpha\beta}$ defined on $[0,T] \times \mathbb{R}^3$, we define the time-slice energy*

$$E_0[F](T) = \sup_{0 \le t \le T} \int_{\Sigma_t} \left(\tau_+^{2s} \left(|\alpha|^2 + |\rho|^2 + |\sigma|^2 \right) + \tau_-^{2s} |\underline{\alpha}|^2 \right) w \, dx, \tag{3.10}$$

the interior (time-slab) energy

$$S_0[F](T) = \int_0^T \int_{\Sigma_t} \left(\tau_+^{2s} |\alpha|^2 + \tau_0^{1+2\iota} \left(\tau_+^{2s} (|\rho|^2 + |\sigma|^2) + \tau_-^{2s} |\underline{\alpha}|^2 \right) \right) w' \, dx \, dt, \tag{3.11}$$

and the conical energy

$$C_0[F](T) = \sup_{u^*} \int_{\{C(u^*)\} \cap \{t \in [0,T]\}} \left(\frac{1}{2} \tau_+^{2s} Q_{\tilde{L}^* L^*} + \frac{1}{2} \tau_-^{2s} Q_{\tilde{L}^* \underline{L}^*} \right) w \, dVC(u^*), \tag{3.12}$$

where Q is the energy-momentum tensor

$$Q[F]_{\alpha\beta} = F_{\alpha\gamma}F_{\beta}^{\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}, \tag{3.13}$$

$\tilde{L}^{\alpha} = \nabla^{\alpha}u^*$, and $C(u^*)$ is the forward light cone $u^* =$ constant.

Then, we have the inequality

$$E_0[F](T) + S_0[F](T) + C_0[F](T) + |q|^2 \lesssim E_0[F](0) + |q|^2 + \|J[F]\|^2_{L^2[w]}. \tag{3.14}$$

We also take the opportunity to define the energy

$$\mathcal{E}_0[F](T) = E_0[F](T) + S_0[F](T) + C_0[F](T), \tag{3.15}$$

as well as the iterated energy

$$\mathcal{E}_k[F](T) = \sum_{\substack{|I|\leq k \\ X\in\mathbb{L}}} \mathcal{E}_0[\mathcal{L}_X^I F](T). \tag{3.16}$$

Remark. *The characteristics of the conical energy C_0 should be mentioned, as there is one important difference between this and the Minkowski case. In particular, we must consider the possibility that $\underline{\alpha}$ terms show up with the wrong sign, due to the behavior of $\tilde{L^*}$. One option to deal with this would be to take exact optical functions and null frames, instead of the approximations we use. We instead leave it as is, as we can later replace this with a more useful positive definite quantity by integrating L^∞ estimates coming purely from the energy E_0 combined with weighted Klainerman-Sobolev-type inequalities. This happens at the cost of more derivatives required on the field. However, in any case, the bottleneck is provided by derivatives of g, so this does not concern us. As a consolation, we have that $C_0[F](T)$ is bounded below by 0 by looking at the case $u^* \geq T$.*

Proof. This is a standard energy estimate based on the fractional Morawetz field used in [16], with the necessary modifications in order to adapt to the modified vector fields and to account for error terms. Our analysis in particular centers around application of the divergence theorem to the momentum density tensor

$$P[F]_{\alpha} = -Q[F]_{\alpha\beta}\overline{K_0^s}^{\beta}\tilde{w}. \tag{3.17}$$

We have the divergence identity

$$\nabla^\alpha P[F]_\alpha = F_{\beta\gamma}\overline{K_0^s}^\beta J^\gamma \tilde{w} - Q[F]_{\alpha\beta}\overline{K_0^s}^{\beta;\alpha}\tilde{w} - Q[F]_{\alpha\beta}\overline{K_0^s}^\beta \nabla^\alpha(\tilde{w}), \tag{3.18}$$

which we apply the divergence theorem to get

$$\int_{\Sigma_T} -\sqrt{|g|}P[F]^0 + \int_{\Sigma_0} \sqrt{|g|}P[F]^0 = \int_0^T \int_{\Sigma_t} \sqrt{|g|}\left(F(\overline{K_0^s}, J)\tilde{w} - Q[F]_{\alpha\beta}\overline{K_0^s}^{\beta;\alpha}\tilde{w} - Q[F]_{\alpha\beta}\overline{K_0^s}^\beta \nabla^\alpha(\tilde{w})\right) dx\, dt. \tag{3.19}$$

We also take the integral along light cones in order to get our conical energy term.

From here we need to show that this gives us what we need. In the interior, we need to show that we have a spacetime integral with a definite sign, along with terms which either have the right sign or are small enough in magnitude that we can subtract them off without issue. Additionally, we need to show that the replacement of $F(\overline{K_0^s}, J)$ with the current norm is valid, which we do via Hölder's inequality. Additionally, we need to show that the time-slice integrals are equivalent to our time-slice energy E_0.

We first recall the deformation tensor identity

$$2(\text{symm})(\nabla\overline{K_0^s}) = \left(\mathcal{L}_{\overline{K_0^s}}g\right). \tag{3.20}$$

Additionally, we consider the commutators

$$[\overline{K_0^s}, L^*] = -\frac{1}{2}L^*(\underline{u}^{*2s})L^* \tag{3.21a}$$

$$[\overline{K_0^s}, \underline{L}^*] = -\frac{1}{2}\underline{L}^*(|u^*|^{2s})\underline{L}^* \tag{3.21b}$$

$$[\overline{K_0^s}, S_j^*] = -\frac{1}{2}\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*}S_j^*. \tag{3.21c}$$

Taking the null decomposition of $\mathcal{L}_{\overline{K_0^s}} g$, and applying the identity (2.8), we get

$$(\mathcal{L}_{\overline{K_0^s}} g)_{L^*L^*} = \overline{K_0^s}(g_{L^*L^*}) + L^*(\underline{u}^{*2s})g_{L^*L^*}, \tag{3.22a}$$

$$(\mathcal{L}_{\overline{K_0^s}} g)_{\underline{L}^*\underline{L}^*} = \overline{K_0^s}(g_{\underline{L}^*\underline{L}^*}) + \underline{L}^*(u^{*2s})g_{\underline{L}^*\underline{L}^*}, \tag{3.22b}$$

$$(\mathcal{L}_{\overline{K_0^s}} g)_{L^*\underline{L}^*} = \overline{K_0^s}(g_{\underline{L}^*L^*}) + 2s(\underline{u}^{*2s-1} + \operatorname{sgn}(u^*)|u^*|^{2s-1})(g_{\underline{L}^*L^*}), \tag{3.22c}$$

$$(\mathcal{L}_{\overline{K_0^s}} g)_{L^*S_j^*} = \overline{K_0^s}(g_{L^*S_j^*}) + \left(2s\underline{u}^{*2s-1} + \frac{1}{2}\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*}\right) g_{L^*S_j^*}, \tag{3.22d}$$

$$(\mathcal{L}_{\overline{K_0^s}} g)_{\underline{L}^*S_j^*} = \overline{K_0^s}(g_{\underline{L}^*S_j^*}) + \left(2s\cdot\operatorname{sgn}(u^*)|u^*|^{2s-1} + \frac{1}{2}\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*}\right) g_{\underline{L}^*S_j^*}, \tag{3.22e}$$

$$(\mathcal{L}_{\overline{K_0^s}} g)_{S_i^*S_j^*} = \overline{K_0^s}(g_{S_i^*S_j^*}) + \frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*} g_{S_i^*S_j^*}, \tag{3.22f}$$

where we have the other terms from symmetry. We establish the bound

$$\frac{1}{2}\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*} \lesssim \tau_+^{2s-1}$$

as follows: first, we write this as

$$\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*} = t^{2s-1}\frac{(1+r^*/t)^{2s} - |1-r^*/t|^{2s}}{r^*/t} = r^{*2s-1}((1+t/r^*)^{2s} - |1-t/r^*|^{2s}). \tag{3.23}$$

exploiting continuity, and taking the t^{2s-1} and r^{*2s-1} terms for $r^*/t \to 0$, $t/r^* \to 0$ respectively, gives us the bound we need.

We take the following auxiliary estimates, which follow from our L^∞ assumptions on the metric:

$$(\mathcal{L}_{\overline{K_0^s}} g)_{\mathcal{L}\mathcal{T}} \lesssim \epsilon\tau_+^{2s-2-\gamma'+\iota}\tau_-^{\gamma'}, \tag{3.24a}$$

$$(\mathcal{L}_{\overline{K_0^s}} g)_{\mathcal{U}\mathcal{U}} \lesssim \tau_+^{2s-1}. \tag{3.24b}$$

We can use the decay estimates on the metric to recast (3.22) as

$$\left|(\mathcal{L}_{\overline{K_0^s}}g)_{L^*L^*}\right| \lesssim \epsilon\tau_+^{2s-2-\gamma'+\iota}\tau_-^{\gamma'} \tag{3.25a}$$

$$\left|(\mathcal{L}_{\overline{K_0^s}}g)_{\underline{L}^*\underline{L}^*}\right| \lesssim \epsilon\tau_+^{2s-2+\iota}, \tag{3.25b}$$

$$\left|(\mathcal{L}_{\overline{K_0^s}}g)_{L^*\underline{L}^*} + 2\left(2s(\underline{u}^{*2s-1} + \operatorname{sgn}(u^*)|u^*|^{2s-1})\right)\right| \lesssim \epsilon\tau_+^{2s-2+\iota}, \tag{3.25c}$$

$$\left|(\mathcal{L}_{\overline{K_0^s}}g)_{L^*S_j^*}\right| \lesssim \epsilon\tau_+^{2s-2-\gamma'+\iota}\tau_-^{\gamma'} \tag{3.25d}$$

$$\left|(\mathcal{L}_{\overline{K_0^s}}g)_{\underline{L}^*S_j^*}\right| \lesssim \epsilon\tau_+^{2s-2+\iota}, \tag{3.25e}$$

$$\left|(\mathcal{L}_{\overline{K_0^s}}g)_{S_i^*S_j^*} - \delta_{ij}\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*}\right| \lesssim \tau_+^{2s-2+\iota}, \tag{3.25f}$$

Raising according to the metric g and applying Lemma 2.5 gives us the component estimates

$$\left|(\text{symm})(\nabla\overline{K_0^s})^{\underline{L}^*\underline{L}^*}\right| \lesssim \epsilon\frac{\tau_-}{\tau_+^{1+\iota}}, \tag{3.26a}$$

$$\left|(\text{symm})(\nabla\overline{K_0^s})^{L^*\underline{L}^*} + \frac{1}{4}\left(2s\underline{u}^{*2s-1} + 2s\operatorname{sgn}(u^*)|u^*|^{2s-1}\right)\right| \lesssim \epsilon\tau_+^{2s-2+\iota}, \tag{3.26b}$$

$$\left|(\text{symm})(\nabla\overline{K_0^s})^{L^*L^*}\right| \lesssim \epsilon\tau_-^{2s-2+\iota}, \tag{3.26c}$$

$$\left|(\text{symm})(\nabla\overline{K_0^s})^{\underline{L}^*S_j^*}\right| \lesssim \epsilon\frac{\tau_-}{\tau_+^{1+\iota}}, \tag{3.26d}$$

$$\left|(\text{symm})(\nabla\overline{K_0^s})^{L^*S_j^*}\right| \lesssim \epsilon\tau_+^{2s-2+\iota}, \tag{3.26e}$$

$$\left|(\text{symm})(\nabla\overline{K_0^s})^{S_i^*S_j^*} - \frac{1}{2}\delta_{ij}\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*}\right| \lesssim \epsilon\tau_+^{2s-2+\iota}. \tag{3.26f}$$

In general, for the components are 0 in Minkowski (i.e., components with only the derivative on the left hand side), we can use(3.24). Additionally, for the nice components we use the inequality $\tau_-^{\gamma'}\tau_+^{2s-2-\gamma'} \lesssim \tau_-\tau_+^{-1-\iota}$. This strong decay in τ_+ turns out to be necessary when we contract this quantity with the EM tensor, and in particular is one of the main reasons we need the nice estimate on good components of the metric.

Before moving forward, we show some intermediate estimates on Q. First, using the shorthand

$$|\not{H}| = |g_{L^*L^*}| + |g_{L^*S^*}| + |g^{\underline{L}^*\underline{L}^*}| + |g^{\underline{L}^*S^*}|, |H| = |H^{\mathcal{U}\mathcal{U}}| + |H_{\mathcal{U}\mathcal{U}}|. \tag{3.27a}$$

we have the estimates

$$\left|F^{\underline{L}^* S_j^*} + \frac{1}{2}\alpha_j\right| \lesssim |H||\alpha| + |\not{H}||F|, \tag{3.28a}$$

$$\left|F^{\underline{L}^* S_j^*} + \frac{1}{2}\alpha_j\right| \lesssim |H||F|, \tag{3.28b}$$

$$\left|F^{S_1^* S_2^*} - \sigma\right| \lesssim |H||\not{F}| + |\not{H}||F|, \tag{3.28c}$$

$$\left|F^{L^* \underline{L}^*} - \frac{1}{2}\rho\right| \lesssim |H||\not{F}| + |\not{H}||F|, \tag{3.28d}$$

all of which directly follow from Lemma (2.5).

Similar reasoning gives us the preliminary estimate

$$\left|F_{\gamma\delta}F^{\gamma\delta} - 2|\sigma|^2 + 2|\rho|^2 + 2\alpha\cdot\underline{\alpha}\right| \lesssim |H||\alpha||F| + |\not{H}||F|^2, \tag{3.29}$$

where $\alpha \cdot \underline{\alpha} = \sum_i \alpha_i \underline{\alpha}_i$,

as well as the subsequent estimates

$$\left|Q_{L^* L^*}[F] - |\alpha|^2\right| \lesssim |H||\alpha|^2 + |\not{H}||\not{F}|^2 + |\not{H}|^2|F|^2, \tag{3.30a}$$

$$\left|Q_{L^* \underline{L}^*}[F] - (|\sigma|^2 + |\rho|^2)\right| \lesssim |H||F||\not{F}| + |\not{H}||F|^2, \tag{3.30b}$$

$$\left|Q_{\underline{L}^* \underline{L}^*}[F] - |\underline{\alpha}|^2\right| \lesssim |H||F|^2, \tag{3.30c}$$

$$\left|\sum_j Q_{S_j^* S_j^*} - (|\sigma|^2 + |\rho|^2)\right| \lesssim |H|(|\alpha||F| + |\not{F}|^2) + |\not{H}||F|^2, \tag{3.30d}$$

$$\left|Q_{S_j^* S_j^*}\right| \lesssim |\not{F}|^2 + |\alpha||F| + |\not{H}||F|^2, \tag{3.30e}$$

$$\left|Q_{S_1^* S_2^*}\right| \lesssim |\alpha||F| + |H|\not{F}|^2 + |\not{H}||F|^2, \tag{3.30f}$$

$$\left|Q_{L^* S_j^*}\right| \lesssim |\alpha||\not{F}| + |\not{H}||F||\not{F}| + |\not{H}|^2|F|^2, \tag{3.30g}$$

$$\left|Q_{\underline{L}^* S_j^*}\right| \lesssim |F||\not{F}| + |H||F|^2. \tag{3.30h}$$

We can now analyze the contribution of the quantity

$$\nabla^\alpha \overline{K_0^s}^\beta Q_{\alpha\beta}.$$

This can be decomposed into a quantity with a definite sign plus a quantity small in magnitude.

Lemma 3.2. *For a given symmetric (0,2)-tensor Q, for the vector field $\overline{K_0^s}$, and a metric g satisfying the metric estimates* (2.14)*, we can write $(\nabla^\alpha \overline{K_0^s}^\beta)Q_{\alpha\beta}$ as follows:*

$$\nabla^\alpha \overline{K_0^s}^\beta Q_{\alpha\beta} = \frac{1}{4}\left(\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*} - \left(2s\underline{u}^{*2s-1} + 2s\, sgn(u^*)|u^*|^{2s-1}\right)\right) Q_{L^*\underline{L}^*} + \\ + \frac{1}{2}\frac{u^{*2s} - |u^*|^{2s}}{r^*} g^{\alpha\beta}Q_{\alpha\beta} + R_1[Q](t,x). \tag{3.31}$$

We have the positivity property

$$\frac{1}{4}\left(\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*} - \left(2s\underline{u}^{*2s-1} + 2s\, sgn(u^*)|u^*|^{2s-1}\right)\right) \geq 0. \tag{3.32}$$

Additionally, the remainder quantity $R_1[Q]$ satisfies the following bounds:

$$|R_1[Q]| \lesssim \frac{\epsilon\tau_-}{\tau_+^{1+2\iota}} Q_{\underline{L}^*\underline{L}^*} + \epsilon\tau_+^{2s-2+\iota}|Q_{\mathcal{T}\mathcal{U}}|. \tag{3.33}$$

If Q is the energy-momentum tensor for some 2-form F, we can bound $R_1[Q](t,x)$ pointwise by

$$|R_1[Q](t,x)| \lesssim \frac{\epsilon}{\tau_+^{1+2\iota}}\left(\tau_+^{2s}(|\alpha[F]|^2 + |\rho[F]|^2 + |\sigma[F]|^2) + \tau_-^{2s}|\underline{\alpha}[F]|^2\right). \tag{3.34}$$

Proof. This is by and large a straightforward result of (3.26) and (3.30).

First, note that in this case we can rewrite the bound (3.34) for $R_1[Q]$ as equivalent to

$$|R_1[Q](t,x)| \lesssim \frac{\epsilon}{\tau_+^{1+2\iota}}(\tau_+^{2s}|\not{F}|^2 + \tau_-^{2s}|F|^2).$$

Integrating this in spacetime gives us the bound

$$\left|\int_0^T \int_{\Sigma_t} R_1[Q](t,x)\tilde{w}\, dx\, dt\right| \lesssim \epsilon S_0[F](T). \tag{3.35}$$

We first consider the first two terms on the right hand side of (3.31) (those other than $R_1[Q]$). These almost trivially follow from the terms on the left hand side of (3.26), with a slight note that error terms coming from the inverse metric g can be treated as follows: first, we define m^* such that

$$m^{*L^*\underline{L}^*} = m^{*\underline{L}^*L^*} = -\frac{1}{2}, \qquad m^{*S_j^*S_k^*} = \delta_{jk},$$

with all other terms in the null decomposition equal to 0. We note that if we replace g with m^* in (3.31), this is simply a restatement of (3.26).

Next, we note that the null decomposition of $(g-m^*)^{\alpha\beta}$ satisfies

$$\frac{\underline{u}^{*2s}-|u^*|^{2s}}{r^*}(g-m^*)^{XY}\lesssim \text{r.h.s.}(3.26), \tag{3.36}$$

in the sense that corresponding components satisfy the estimates. We can fold our remaining analysis into our bounds on remaining error terms coming from the right hand side of (3.26).

The positivity property (3.32) is equivalent to Lemma 3.2 in [16], by making the appropriate coordinate substitution.

We now look at the error terms. We look at the error terms coming from the deformation tensor, corresponding to the right hand sign of (3.26). These are easy to deal with, mostly using the decomposition

$$\epsilon\tau_+^{2s-2+\iota}|F||\not{F}|\lesssim \epsilon\left(\tau_+^{2s-1-2\iota}|\not{F}|^2+\tau_+^{2s-3+4\iota}|F|^2\right), \tag{3.37}$$

and noting that $2s-3+4\iota<-1-2\iota$ under the condition that $s+4\iota<1$.

In all terms worse than this (containing $|F|^2$), we have either an extra power $\tau_-^{-1+2\iota}$ or extra decay in τ_+ coming from the metric, both of which are easily contained in the right hand side of (3.34). Now we consider error terms appearing on the right hand side of (3.30) We note that we only care when these are paired with terms on the left hand side of (3.26). We have uniform bounds like

$$\epsilon\tau_+^{2s-2+\iota}|F||\not{F}|, \tag{3.38}$$

which can be treated in the same way. □

Now we look at the terms where the derivative falls on the metric. We decompose

$$\nabla^\alpha(w)=g^{\alpha\underline{L}^*}\underline{L}^*(\tilde{w})+g^{\alpha L^*}L^*(\tilde{w}). \tag{3.39}$$

We can use the estimates (3.4) to find decompose this as follows:

$$\nabla^\alpha(w)=-\frac{1}{2}\underline{L}^*(\tilde{w})L^{*\alpha}-\frac{1}{2}L^*(\tilde{w})\underline{L}^{*\alpha}+R_2^{L^*}L^{*\alpha}+R_2^{S_j^*}S_j^{*\alpha}+R_2^{\underline{L}^*}\underline{L}^{*\alpha}, \tag{3.40}$$

where R_2 can be thought of as a remainder tensor which depends on the deviation of the metric from Minkowski. This follows from the remaining null decomposition of g. We first note that

$$-Q_{\overline{K_0^s}\alpha}\left(-\frac{1}{2}\underline{L}^*(\tilde{w})L^{*\alpha}-\frac{1}{2}L^*(\tilde{w})\underline{L}^{*\alpha}\right)\leq -cS_0[F](T), \tag{3.41}$$

for some positive constant c depending only on the decay constants δ, ι.

In order for this to meaningful we must establish bounds on the remainder tensor R_2^α, or equivalently its components R_2^X. Taking the null decomposition of g gives us:

$$|R_2^{L^*}|\lesssim \epsilon\tau_+^{-1+\iota}w', \qquad |R_2^{\underline{L}^*}|+|R_2^{S_j^*}|\lesssim \epsilon\tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}w' \tag{3.42}$$

Therefore, the total set of remainder terms corresponding to R_2 can be bounded pointwise by

$$\epsilon\left(\tau_+^{-1+\iota}|Q_{\overline{K_0^s}L^*}|+\tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}(|Q_{\overline{K_0^s}S_j^*}|+|Q_{\overline{K_0^s}\underline{L}^*}|)\right)w'. \tag{3.43}$$

Using the estimates (3.30) as well as the identities

$$\epsilon^{-1}|H|^2+|\not{H}|\lesssim \epsilon\tau_0^{2s}\leq \epsilon\tau_0^{1+2\iota}$$

gives us

$$\int_0^T\int_{\Sigma_t}|Q[F]_{\overline{K_0^s}R}|\,dx\,dt\lesssim \epsilon S_0[F](T). \tag{3.44}$$

We now turn our attention to the boundary conditions. It suffices to show that

$$\int_{\Sigma_t}-\sqrt{|g|}g^{0\alpha}P[F]_\alpha\approx E_0[F](t). \tag{3.45}$$

For this we only need the pointwise estimate

$$|-g^{0\alpha}P[F]_\alpha-P[F]_0|\lesssim \epsilon|P[F]_0|,$$

as we can make $\sqrt{|g|}$ arbitrarily close to 1.

This is fortunately straightforward to show, as all of the worst components of $-g^{0\alpha}P[F]_\alpha$ are already in $P[F]_0$. We can define the decomposition $-g^{0\alpha}=\delta_0^\alpha+{}^tH\alpha$, where components in the null decomposition of

${}^tH$ satisfy

$$ {}^tH^X \lesssim \epsilon\tau_+^{-1+\iota}. $$

The L^* and $\underline{L}^*$ components of $P[F]_\alpha {}^tH^\alpha$ are easy to bound by $P[F]_0$, as the null decomposition can be written as a linear combination of these terms. Likewise, expanding $P[F]_{B_j}$ out using (3.30) easily gives us the rest of our bound. (3.45) therefore holds.

We next consider the terms on the light cone. Again, we note that we do not have the nice positivity property in this term. However, we note that we have the integral

$$ \int_{C(u^*)} -\nabla^\alpha(u^*)\sqrt{|g|}P[F]_\alpha \, dV(C), \tag{3.46} $$

which we can approximate in a similar fashion to the weight. Note here that $dV(C)$ is the area element with respect to the Minkowski metric. We have, in particular,

$$ \left| -\nabla^\alpha(u^*)P[F]_\alpha - \frac{1}{2}P[F]_{L^*} \right| \lesssim \epsilon\tau_+^{-1+\iota}P[F]_{L^*} + \tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}\left(|P_{S_1^*}| + |P_{S_2^*}| + |P_{\underline{L}^*}|\right). \tag{3.47} $$

The first term on the right appears in our light cone energy. However, even this contains a bad component of F. We can instead take a component estimate

$$ \left| -\nabla^\alpha(u^*)P[F]_\alpha - \frac{1}{2}P[F]_{L^*} \right| \lesssim \epsilon\left(\tau_+^{2s-1+\iota}|\alpha|^2 + \tau_-^{2s}\tau_+^{-1+\iota}|\not{F}|^2 + \tau_-^{2s+\gamma'}\tau_+^{-1-\gamma'+\iota}|F|^2\right)\tilde{w}. \tag{3.48} $$

The first two appear with a definite sign in $P[F]_{L^*}$, while we need to deal with the last term in a different way. We will essentially establish an improved energy

$$ C_0^*[F](T) = \sup_{u^*}\int_{\{C(u^*)\}\cap\{t\in[0,T]\}} \left(\tau_+^{2s}|\alpha|^2 + \tau_-^{2s}(|\sigma|^2 + |\rho|^2)\right) w\, dVC(u^*), \tag{3.49} $$

and we will show the estimate

$$ |C_0^*[F](T)| \lesssim |C_0[F](T)| + |E_2[F](T)|. \tag{3.50} $$

We note that we do not have a definite sign for error terms on the right (or even in bad components of $P[F]_{L^*}$); in particular, in this integral, terms like $\underline{\alpha}$ only appear as error terms. This will be handled later by integrating certain L^∞ estimates which depend only on our time-slice and interior energies.

We are now able to put everything together. We first apply the divergence theorem (in Minkowski space) on the quantity $-\sqrt{|g|}P[F]^\alpha$, over the time slab $[0,T]\times\mathbb{R}^3$. We recall

$$\int_{\Sigma_T} -\sqrt{|g|}P[F]^0 + \int_{\Sigma_0}\sqrt{|g|}P[F]^0 = \int_0^T\int_{\Sigma_t}\sqrt{|g|}\left(F(\overline{K_0^s},J)\tilde{w} - Q[F]_{\alpha\beta}\overline{K_0^s}^{\beta;\alpha}\tilde{w} - Q[F]_{\alpha\beta}\overline{K_0^s}^{\beta}\nabla^\alpha(\tilde{w})\right)\,dx\,dt.$$

Applying , Lemma 3.2, as well as equations (3.45), (3.40), and (3.44), moving all negative definite terms coming from (3.40) to the left, and disregarding those from Lemma 3.2 gives us

$$E_0[F](T) + S_0[F](T) \lesssim E_0[F](0) + \int_0^T\int_{\Sigma_t}\left(\left|F(\overline{K_0^s},J)\right|w + \epsilon S_0[F](T)\right)\,dx\,dt. \tag{3.51}$$

We note that we can move the last term on the right over to the left without losing anything. We now consider the remaining term on the right hand side, containing the current. We have the inequalities

$$|J_L g^{L\beta}F_{\overline{K_0^s}\beta}| \lesssim J_L\left(\tau_+^{2s}|\not{F}| + \tau_-^{2s}|H||F|\right) \tag{3.52a}$$

$$|J_{S_j^*} g^{S_j^*\beta}F_{\overline{K_0^s}\beta}| \lesssim J_{S_j^*}\left(\tau_+^{2s}|\alpha| + \tau_-^{2s}|F|\right) \tag{3.52b}$$

$$|J_{\underline{L}^*} g^{\underline{L}^*\beta}F_{\overline{K_0^s}\beta}| \lesssim J_{\underline{L}^*}\left(\tau_-^{2s}|\not{F}| + \tau_-^{2s}|\not{H}||F|\right) \tag{3.52c}$$

Application of Hölder's inequality, keeping in mind the current norm (3.9), gives us

$$\int_0^T\int_{\Sigma_t}|F(\overline{K_0^s},J)|w\,dx\,dt \lesssim S_0[F](T)^{1/2}\,\|J\|_{L^2[w]}\,. \tag{3.53}$$

By Cauchy-Schwarz, we have that this can be bounded by

$$C^{-1}S_0[F](T) + C\,\|J\|^2_{L^2[w]}\,.$$

For some C independent of ϵ, we can move the $S_0[F](T)$ term over to the left hand side.

Finally, in order to include the conical energy, we repeat the divergence theorem on regions of the form

$$([0,T]\times\mathbb{R}^2)\cap\{t-r^*\le c\}.$$

The maximum integral over the reduced light cone gives us our conical energy. $\square$

We must be careful here, since this theorem is not useful for general electromagnetic fields. In particular, we require initial decay faster than r^{-2}, which is not possible in the presence of a charge, due to elliptic considerations of the initial data. Our solution here is to divide the field F into two parts: $\overline{F}$, an explicitly defined quantity coming from the charge whose behavior is straightforwardly bounded, and the remainder quantity $\tilde{F}$, which has better spatial decay. We will more clearly define these quantities when we look at the commutators.

4 The L^2 Estimate: ϕ

Now we consider the L^2 estimate for ϕ. We can again consider this an analogue to the conformal Morawetz estimate on the Schwarzschild exterior, as the one by Blue and Sterbenz in [3].

Before proceeding, we give some geometric motivation. In the Minkowski metric, we consider the conformal Killing field

$$K_0 = \underline{u}^2(\partial_t + \partial_r) + u^2(\partial_t - \partial_r).$$

In the case of a trace-free energy-momentum tensor, for instance $Q[F]$, one can easily construct an energy estimate using this field. However, in general, we have a term containing the trace of the energy-momentum tensor, which has no sign. One method to deal with this is to take analysis on the metric

$$^{I}m = \frac{1}{r^2}m,$$

in which K_0 is indeed Killing away from the spatial origin. This has in [16] been augmented with the estimate

$$^{II}m = \frac{1}{(u\underline{u})^2}m;$$

however, this is more difficult to adapt to a general metric, due to its singular behavior along the light cones (as compared with ^{I}m, which is singular in the deep interior, where we can more easily model g by the Minkowski metric). Instead, we use a Hardy-type estimate which gives us almost the same results, requiring slightly more decay in ϕ.

In any Lorentzian metric we have the commutator

$$\frac{1}{r^*}\left[\Box_g^{\mathbb{C}}, r^*\cdot\right]\phi = \frac{r^*\phi}{r^{*2}}\Box_g(r^*) + \frac{2}{r^{*2}}\nabla^\alpha(r^*)D_\alpha(r^*\phi) - \frac{2}{r^{*2}}\nabla^\alpha(r^*)\nabla_\alpha(r^*)\phi \tag{4.1}$$
$$= r^*\phi\Box_g\left(\frac{1}{r^*}\right) - 2\nabla^\alpha\left(\frac{1}{r^*}\right)D_\alpha(r^*\phi).$$

The singular behavior near the spatial origin is of concern, as in a general metric we can no longer consider $\frac{1}{r^*}$ as a solution of the wave equation for general metrics. One solution is to base our geometric analysis on some first-order reduced metric m^*, which is equal to m near the spatial origin and behaves like our exterior Schwarzschild metric in the extended exterior. We define the metric

$$m^* = -\left(1 - \frac{M\chi}{1+r+M}\right)dt^2 + \left(1 - \frac{M\chi}{1+r+M}\right)^{-1}dr^2 + \left(1 - \frac{M\chi}{1+r+M}\right)^{-1}r^2dS^2. \tag{4.2}$$

This is of course equal to the Minkowski metric in the interior. In the exterior, this is equal to

$$-\left(1 - \frac{M}{1+r+M}\right)dt^2 + \left(1 + \frac{M}{1+r}\right)dr^2 + \left(1 + \frac{M}{1+r}\right)r^2\,dS^2.$$

It is therefore close to $\tilde{m}$, in the sense that they have the same exterior decay up to order $M\tau_+^{-2}$.

In this metric, we define the time-slice energy

$$E_0[\phi](T) = \sup_{0\le t\le T}\int_{\Sigma_t}\left(\tau_+^{2s}\left(\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + |\not{D}\phi|^2 + \left|\frac{\phi}{r^*}\right|^2\right) + \tau_-^{2s}|D_{\underline{L}^*}\phi|^2\right)w\,dx. \tag{4.3}$$

It follows from Lemma 9.8 that this energy is equivalent to

$$\sup_{0\le t\le T}\int_{\Sigma_t}\left(\tau_+^{2s}\left(\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + |\not{D}\phi|^2\right) + \tau_-^{2s}\left|\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|^2\right)w\,dx.$$

Likewise, we have the interior spacetime energy

$$S_0[\phi](T) = \int_0^T\int_{\Sigma_t}\left(\tau_+^{2s}\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + \tau_0^{1+2\iota}\left(\tau_+^{2s}\left(|\not{D}\phi|^2\right) + \tau_-^{2s}\left(|D_{\underline{L}^*}\phi|^2 + \left|\frac{\phi}{r^*}\right|^2\right)\right)\right)w'\,dx\,dt. \tag{4.4}$$

We note that this is weaker than the corresponding term in [16], since the aforementioned estimate has $(\phi/r)^2$ with a weight of τ_+^{2s} instead of τ_-^{2s}. However, for the interior estimate we can again use a Poincare-type estimate, in particular (9.29), to replace the weight τ_-^{2s} with τ_+^{2s} as long as we have the inequality $s+\delta>1$. This is worse than the decay required in [16]. Note that this condition only comes from the charged portion of the field, and as such, in the charge-free case we have only the condition $s > 1/2$. This follows from the use of (4.11c), which provides the exact same decay along the light cone as the estimate in [16], with a slightly worse weight in the far exterior.

Finally, we have the light-cone energy

$$C_0[\phi](T) = \sup_{u^*} \int_{C(u^*)} Q(\nabla^\alpha_{m^*} u^*, \overline{K^s_0}) w \, dC(u^*), \tag{4.5}$$

where Q is defined in equation (4.15) and ∇_{m^*} is the covariant derivative with respect to the modified metric m^* as defined in equation (4.2). In the rest of this subsection all analysis will be conducted with respect to the metric m^*.

Note that the energy C_0, analogous to the corresponding conical energy for F, is not positive definite in ϕ, though it is nonnegative. However, we later show that for fewer derivatives of ϕ it is indeed positive definite up to a small quantity scaling with energies on higher numbers of derivatives, combined with the usual L^∞ norms on the metric. We can in particular define a reduced conical energy

$$C^*_0[\phi](T) = \sup_{u^*} \int_{C(u^*)} \left(\left| \tau_+^{2s} \frac{D_{L^*}(r^*\phi)}{r^*} \right|^2 + \tau_-^{2s} \left| D_{S^*}\phi \right|^2 + \tau_+^{2s} \tau_0^2 \left| \frac{\phi}{r^*} \right|^2 \right) w \, dVC(u^*).$$

We take the combined energy

$$\mathcal{E}_0[\phi](T) = E_0[\phi](T) + S_0[\phi](T) + C_0[\phi](T) \tag{4.6}$$

In general, we define

$$\mathcal{E}_k[\phi](T) = \sum_{\substack{|I| \le k \\ X \in \mathbb{L}}} E_0[D^I_X \phi](T). \tag{4.7}$$

We can define the analogous quantities $E_k[\phi](T)$, $S_k[\phi](T)$, $C_k[\phi](T)$ similarly. With this notation we state a useful estimate which we will prove later: We will show later the inequality

$$C^*_0[\phi](T) \lesssim C_0[\phi](T) + \epsilon E_2[\phi](T).$$

Here, we assume the energy is less than 1, so we can write e.g. $\mathcal{E}_0[\phi](T)^m \le \mathcal{E}_0[\phi](T)$ for $m \ge 1$.

We take the opportunity here to define the full energy

$$\mathcal{E}_k(T) = \mathcal{E}_k[\phi] + \mathcal{E}_k[\widetilde{F}] + |q|^2, \tag{4.8}$$

where $\widetilde{F}$ is the charge-free portion of the energy F. When there is no ambiguity we write $\mathcal{E}_k$.

We can now state the main theorem:

Theorem 4.1. *For a given function ϕ with sufficient decay, we have*

$$\mathcal{E}_0[\phi](T) \lesssim \mathcal{E}[\phi](0) + \left(|q| + \left\|\widetilde{F}\right\|^{red}_{L^\infty[w]}\right)\mathcal{E}_0[\phi](T) + \left\|\tau_+^s \tau_-^{1/2}(\Box_g^{\mathbb{C}}\phi)w_\iota^{1/2}\right\|_2^2, \tag{4.9}$$

where we define the reduced weighted L^∞ norm

$$\left\|\widetilde{F}\right\|^{red}_{L^\infty[w]} = \tau_+\tau_-^{1/2+s}\underline{\alpha}[\widetilde{F}]w^{1/2} + \tau_+^{1+s}\tau_-^{1/2}(|\rho[\widetilde{F}]| + |\sigma[\widetilde{F}]|)w^{1/2} + \tau_+^{3/2+s}|\alpha[\widetilde{F}]|w^{1/2}. \tag{4.10}$$

Proof. We deal with this in two parts: first, we use the estimate (4.27), which adapts the estimate in the Minkowski case to our first-order modified metric. We then handle the error terms in the metric using Lemma 4.3. The second term on the right hand side follows from Hölder's inequality using estimate (4.11f). $\square$

We note the following inequalities which will be used many times in the future:

$$\left\|\tau_+^{-1/2-\iota}\tau_-^s|D\phi|w^{1/2}\right\|_2^2 \lesssim E_0[\phi](T) \tag{4.11a}$$

$$\left\|\tau_+^{s-1-\gamma'/2+\iota/2}\tau_-^{\gamma'/2}|D\phi|w^{1/2}\right\|_2^2 \lesssim E_0[\phi](T) \tag{4.11b}$$

$$\left\|\tau_+^{s-3/2-\iota}|\phi|w^{1/2}\right\|_2^2 \lesssim E_0[\phi](T) \tag{4.11c}$$

$$\left\|\tau_+^{s-1/2-\iota}|\overline{D}\phi|w^{1/2}\right\|_2^2 \lesssim E_0[\phi](T) \tag{4.11d}$$

$$\left\|\tau_+^{2s-2+\iota}|\overline{D}\phi||D\phi|w\right\|_1 \lesssim E_0[\phi](T) \tag{4.11e}$$

$$\left\|\tau_+^{-s}\frac{D_{\overline{K_0^s}}(r^*\phi)}{r^*}(w')^{1/2}\right\|_2^2 \lesssim S_0[\phi](T). \tag{4.11f}$$

The inequalities (4.11a)-(4.11e) follow from the inequality, for $\iota > 0$,

$$\int_0^T \frac{E_0[\phi](t)}{(1+t)^{1+2\iota}}\,dt \lesssim_\iota E_0[\phi](T). \tag{4.12}$$

Additionally, for (4.11b), we use $s - 1 - \gamma'/2 + \iota < -\frac{1}{2} - \iota$, $\gamma'/2 < 1/2 < s$, and (4.11e) follows from an application of Hölder's inequality, combined with the inequality $s - 3/2 + 2\iota \leq -1/2 - \iota$. Inequality (4.11f) comes from our spacetime energy norm. In all cases we can alternatively replace $E_0(t)$ with $S_0(t)$ on the right hand side with no issue, using the identity $w \lesssim \tau_-^{1+2\iota}w'$. These will in general be useful when we need less precise estimates.

It follows from (4.11b) and (4.11e) that

$$\left\|\tau_+^{2s-1} H^{\gamma\delta} D_\gamma\phi\overline{D_\delta\phi} w\right\|_1 \lesssim \epsilon E_0[\phi](T), \tag{4.13}$$

which will be used many times later.

We recall that by the harmonic gauge condition,

$$\nabla_g^\alpha D_\alpha\phi = g^{\alpha\beta}\partial_\alpha D_\beta\phi. \tag{4.14}$$

For the sake of convenience we define

$$R = \left(1 - \frac{M\chi}{1+r+M}\right)\partial_r.$$

It is easy to see that $R = \partial_{r^*}$ in the region where $\chi' = 0$, and can be bounded by

$$R(r^*) - 1 \lesssim M\tau_+^{-1+\iota}$$

in the region where $\chi' \neq 0$.

Thus, the difference between this and the metric $\tilde{m}$ scales with M and and decays like τ_+^{-2} in the extended exterior.

We consider the energy-momentum tensor

$$Q^*[\phi]_{\alpha\beta} = \frac{1}{r^{*2}}\mathfrak{R}\left(D_\alpha(r^*\phi)\overline{D_\beta(r^*\phi)} - \frac{1}{2}m^*_{\alpha\beta}m^{*\gamma\delta}D_\gamma(r^*\phi)\overline{D_\delta(r^*\phi)}\right). \tag{4.15}$$

It follows that

$$\nabla^\alpha Q^*[\phi]_{\alpha\beta} = -\frac{2\nabla^\alpha(r^*)}{r^*}Q^*[\phi]_{\alpha\beta} + \frac{1}{r^{*2}}\mathfrak{R}\left(\Box^{\mathbb{C}}_{m^*}(r^*\phi)\overline{D_\beta(r^*\phi)}\right) + \frac{1}{r^{*2}}\mathfrak{I}\left(r^*\phi\overline{D^\alpha(r^*\phi)}\right)F_{\alpha\beta}. \tag{4.16}$$

We use formula (4.1) to replace the second term with

$$\frac{1}{r^{*2}}\mathfrak{R}\left(\left(r^*\Box^{\mathbb{C}}_{m^*}\phi + r^{*2}\phi\Box_{m^*}\left(\frac{1}{r^*}\right) + \frac{2\nabla^\alpha(r^*)}{r^*}D_\alpha(r^*\phi)\right)\overline{D_\beta(r^*\phi)}\right).$$

Combining these and taking the necessary cancellations gives us

$$\begin{aligned}\nabla^\alpha Q^*[\phi]_{\alpha\beta} &= \frac{\nabla^\alpha(r^*)}{r^{*3}} m^*_{\alpha\beta} D_\gamma(r^*\phi)\overline{D^\gamma(r^*\phi)}+ \\ &\quad + \Re\left(\left(\Box^{\mathbb{C}}_{m^*}\phi + r^*\phi\Box_{m^*}\left(\frac{1}{r^*}\right)\right)\frac{\overline{D_\beta(r^*\phi)}}{r^*}\right) + \Im\left(\phi\frac{\overline{D^\alpha(r^*\phi)}}{r^*}\right)F_{\alpha\beta}. \\ &= P^1_\beta + P^2_\beta + P^3_\beta.\end{aligned} \tag{4.17}$$

We note that P^2 consists of two terms, one of which will be handled when we consider the metric perturbation H, along with commutators, and the other of which is small. In particular, we have the estimate

$$\Box_{m^*}\left(\frac{1}{r^*}\right) \lesssim M\tau_+^{-4+\iota}. \tag{4.18}$$

This is 0 in the Minkowski case, so we can think of this as the set of error terms coming from the metric (of order $M\tau_+^{-1+\iota}$) multiplied by a second derivative which decays like τ_+^{-3}. It follows that

$$\begin{aligned}\left\| |r^*\phi\Box_{m^*}\left(\frac{1}{r^*}\right)\overline{D_{\overline{K_0^s}}(r^*\phi)}w\right\|_1 &\lesssim M\left\|\tau_+^{-3/2+s+2\iota}\frac{\phi}{\tau_+}w^{1/2}\right\|_2\left\|\tau_+^{-s-1/2-\iota}\frac{D_{\overline{K_0^s}}(r^*\phi)}{r^*}w^{1/2}\right\|_2. \\ &\lesssim M\mathcal{E}_0[\phi](T).\end{aligned} \tag{4.19}$$

We have obtained the last line from the inequalities (4.11c) and (4.11f).

For P^3, we split F into its charge and remainder terms. First, for the charge terms, we have the estimate

$$\left\|\phi\frac{\overline{D^\alpha(r^*\phi)}}{r^*}\overline{F}_{\alpha\beta}\overline{K_0^s}^\beta w\right\|_1 \lesssim \left\|\tau_+^{s-2}\tau_-\phi(w')^{1/2}\right\|_2\left\|\tau_+^{2-s}\frac{D^\alpha(r^*\phi)}{r^*}\overline{F}_{\alpha\beta}\overline{K_0^s}^\beta(w')^{1/2}\right\|_2, \tag{4.20}$$

where we use Cauchy-Schwartz along with the identity $w \le \tau\_w'$ in the exterior, where $\overline{F}$ is supported. The first term is bounded by

$$\left\|\tau_+^{-1}\tau_-^s\phi(w')^{1/2}\right\|_2,$$

which is contained in our norm, using the exterior version of Lemma 9.10 along with the inequality $s+\delta > 1$.

We can expand to get the estimate

$$\left|\frac{D^\alpha(r^*\phi)}{r^*}\overline{F}_{\alpha\beta}\overline{K_0^s}^\beta\right| \lesssim |q|\left(\left|\tau_+^{2s-2}\frac{D_{L^*}(r^*\phi)}{r^*}\right| + \epsilon\left|\tau_-^{2s}\tau_+^{-3+\iota}\frac{D_{S^*}(r^*\phi)}{r^*}\right| + \left|\tau_-^{2s}\tau_+^{-2}\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|\right) \tag{4.21}$$

We can plug these in to (4.20) to see that

$$\left\|\phi\frac{\overline{D^\alpha(r^*\phi)}}{r^*}\overline{F}_{\alpha\beta}\overline{K_0^s}^\beta w\right\|_1 \lesssim |q|\mathcal{E}_0[\phi](T). \tag{4.22}$$

Note that we have some room along the light cone with this estimate, but we require very sharp spatial decay.

Now we consider the remainder terms, where we have less room along the light cone, but better spatial decay. We take the estimate

$$\left\|\phi\frac{\overline{D^\alpha(r^*\phi)}}{r^*}\widetilde{F}_{\alpha\beta}\overline{K_0^s}^\beta w\right\|_1 \lesssim \left\|\tau_+^{s-3/2-\iota}\phi w^{1/2}\right\|_2 \left\|\tau_+^{3/2+\iota-s}\frac{D^\alpha(r^*\phi)}{r^*}\widetilde{F}_{\alpha\beta}\overline{K_0^s}^\beta w^{1/2}\right\|_2. \tag{4.23}$$

We have the bound

$$\left|\frac{D^\alpha(r^*\phi)}{r^*}\widetilde{F}_{\alpha\beta}\overline{K_0^s}^\beta\right| \lesssim \left\|\widetilde{F}\right\|^{red}_{L^\infty[w]}\left(\left|\tau_+^{-1+s}\tau_-^{-1/2}\frac{D_{L^*}(r^*\phi)}{r^*}\right| + \left|\tau_+^{s-3/2}\frac{D_{S^*}(r^*\phi)}{r^*}\right| + \left|\tau_+^{-1-s}\tau_-^{2s-1/2}\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|\right) \tag{4.24}$$

We note that the terms coming from the error in the metric can be disregarded, as they decay quickly enough that they do not introduce any new decay requirements. We can plug each of these into (4.23), and by the inequality $1/2 + 2\iota < s$, it is easy to see that these fall within our desired bounds.

Next we look at the deformation tensor with respect to the metric m^*. This fortunately satisfies the same estimates as (3.26), and Lemma 3.2 follows. Thus,

$$(\nabla^\alpha\overline{K_0^s}^\beta)Q^*[\phi]_{\alpha\beta} = \frac{1}{2}\left(\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*}\right)(\text{tr } Q) + R_1[Q^*](t,x) + f(t,x)Q^*_{L^*\underline{L}^*}, = P^4 + P^5 + P^6. \tag{4.25}$$

We note that F is positive up to terms which can be controlled in magnitude, similarly to P^5. We consider $P^1 + P^4$. We have

$$m^*_{\alpha\beta}\nabla^\alpha(r^*)\overline{K_0^s}^\beta - \frac{1}{2}\left(\frac{\underline{u}^{*2s} - |u^*|^{2s}}{r^*}\right) = M \cdot O(\tau_+^{2s-2+\iota}), \tag{4.26}$$

which follows as usual from the fact that we can regard the difference as error terms scaling with the metric and which vanish for $r < \frac{t}{2}$. It follows that the quantity $P^1 + P^4$ satisfies the same estimates as $R_1[Q^*]$. We are now ready to show the first part of our energy estimate.

Lemma 4.2. *Given a function ϕ, for sufficiently small M we have the inequality*

$$\mathcal{E}_0[\phi](T) \lesssim \mathcal{E}_0[\phi](0) + \left(|q| + \left\|\widetilde{F}\right\|^{red}_{L^\infty[w]}\right)\mathcal{E}_0[\phi](T) + \left|\int_{[0,T]\times\Sigma_t} \Box^{\mathbb{C}}_{m^*}\phi \frac{\overline{D_{\overline{K^s_0}}(r^*\phi)}}{r^*} w\right| + \\ + \left|\int_{[0,T]\times\Sigma_t\setminus\{C_{max}\}} \Box^{\mathbb{C}}_{m^*}\phi \frac{\overline{D_{\overline{K^s_0}}(r^*\phi)}}{r^*} w\right| \tag{4.27}$$

The domain of the last integral, containing $\{C_{max}\}$, is the integral outside the light cone corresponding to our C_0 energy.

It is important that we use the absolute value in our last two integrals instead of the L^1 norm as it is necessary for certain bounds later on.

Proof. This follows from an application of the divergence theorem over time slabs (and over the reduced slabs $[0,T]\times\{u^* \leq C\}$.) We have that

$$-\nabla^\alpha\left(Q^*[\phi]_{\alpha\beta}\overline{K^s_0}^\beta\tilde{w}\right) = -P^1_{\overline{K^s_0}}\tilde{w} - P^2_{\overline{K^s_0}}\tilde{w} - P^3_{\overline{K^s_0}}\tilde{w} - P^4\tilde{w} - P^5\tilde{w} - P^6\tilde{w} - Q^*[\phi](\nabla\tilde{w}, \overline{K^s_0}). \tag{4.28}$$

We recall that

$$|P^1_{\overline{K^s_0}} + P^4| + |P^5| \lesssim M\tau_+^{2s-2-\iota}\mathrm{tr}(Q^*).$$

Similarly,

$$-P^6\tilde{w} \lesssim M\tau_+^{2s-2-\iota}\mathrm{tr}((Q^*).$$

Note that this is not true in magnitude. We combine these to get

$$\int_0^T\int_{\Sigma_t}(-P^1_{\overline{K^s_0}}\tilde{w} - P_4\tilde{w} - P_5\tilde{w} - P_6\tilde{w}) \lesssim M\mathcal{E}_0[\phi](T). \tag{4.29}$$

Again, this is true in sign, not in magnitude.

Next, $P^2_{\overline{K^s_0}}$ and $P^3_{\overline{K^s_0}}$ are the terms appearing on the right hand side of (4.27), up to a term which is also bounded by the energy (by magnitude).

When the derivative falls on $\tilde{w}$, we need to show that this is equivalent to $S_0[\phi]$. We have the null decomposition

$$\nabla^\alpha(\tilde{w}) = L^{*\alpha}m^{*L^*\underline{L}^*}\underline{L}^*(\tilde{w}) + \underline{L}^{*\alpha}m^{*\underline{L}^*L^*}L^*(\tilde{w}) + L^{*\alpha}m^{*L^*L^*}L^*(\tilde{w}) + \underline{L}^{*\alpha}m^{*\underline{L}^*\underline{L}^*}\underline{L}^*(\tilde{w}) \tag{4.30}$$

The first term is equivalent to $L^{*\alpha}w'$, and the second term is equivalent to $\underline{L}^{*\alpha}\tau_0^{1+2\iota}w'$. The final two terms are supported in the region where $\tau_+ \approx \tau_-$. We look at the corresponding terms:

$$Q^*[\phi](L^*, \overline{K_0^s})w' + Q^*[\phi](\underline{L}^*, \overline{K_0^s})\tau_0^{1+2\iota}w' \geq C^{-1}\left(\tau_+^{2s}Q^*_{L^*L^*} + \tau_0^{1+2\iota}\tau_+^{2s}Q_{\underline{L}^*L^*} + \tau_0^{1+2\iota}\tau_-^{2s}Q_{\underline{L}^*\underline{L}^*}\right).$$

for some positive C. Expanding, integrating in spacetime, and taking our Hardy estimate (9.8) on time slices, this is equivalent to $S_0[\phi]$. The remainder terms are easily bounded by $MS_0[\phi](T)$.

Now we look at the boundary. We need to show that

$$\int_{\Sigma_t} -(\nabla^\alpha t)Q^*[\phi]_{\alpha\overline{K_0^s}}\tilde{w} \approx \int_{\Sigma_t}\left(\tau_+^{2s}\left(\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + |\not{D}\phi|^2\right) + \tau_-^{2s}\left|\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|^2\right) w\,dx. \tag{4.31}$$

The integrand on the left is equivalent to

$$Q^*[\phi](\partial_t, \overline{K_0^s})w,$$

so we take our analysis on that quantity. We can replace ∂_t with ∂_{t^*}, noting that their difference is in the direction of ∂_{r^*} and decays like $M\tau_0^{2s}\tau_+^{-1+\iota}$, so in particular we can bound the terms coming from the difference by $ME_0[\phi]$. Therefore, we can take the null decomposition

$$\frac{1}{4}\int_{\Sigma_t}(1+\underline{u}^{*2s})Q^*_{L^*L^*} + (2+\underline{u}^{*2s}+|u^*|^{2s})Q^*_{L^*\underline{L}^*} + (1+|u^*|^{2s})Q^*_{\underline{L}^*\underline{L}^*}\,dx. \tag{4.32}$$

By our usual estimates on Q^* this is equivalent to

$$\int_{\Sigma_t}\left(\tau_+^{2s}\left(\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + |\not{D}\phi|^2\right) + \tau_-^{2s}\left|\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|^2\right) w\,dx. \tag{4.33}$$

Finally, we look at the C_0 norm. We note that

$$|(\nabla u^*)^{L^*} - 1| \lesssim M\tau_+^{-2}$$
$$|(\nabla u^*)^{\underline{L}^*}| \lesssim M\tau_+^{-2}$$

where the latter term is supported where $\tau_+ \approx \tau_-$. Therefore, the C_0 energy can be written as

$$C_0[\phi] \approx \sup_{u^*}\int_{C(u^*)}\left(\tau_+^{2s}\left|\frac{D_{L^*}(r^*\phi)}{r^*}\right|^2 + \tau_-^{2s}|\not{D}\phi|^2\right)w + \tau_+^s O(M\tau_+^{-2})\left|\frac{D_{\underline{L}^*}(r^*\phi)}{r^*}\right|^2 w\,dC(u^*). \tag{4.34}$$

The last term is supported away from the origin, so we can bound it above by

$$\big||D_{\underline{L}^*}\phi| + \left|\frac{\phi}{\tau_+}\right|$$

without issue. $\square$

Lemma 4.3. *Defining*

$$C_{err}[\phi] = \sup_{u^*} \int_{C_{u^*}} \partial_\beta(u^*)\tilde{Q}[\phi]^\beta_{\overline{K_0^s}} w, \tag{4.35}$$

where $\partial_\beta(u^*)\tilde{Q}[\phi]^\beta_{\overline{K_0^s}}$ *is defined in* (4.45)*, The following inequality holds for any function* ϕ*:*

$$\left|\int_{[0,T]\times\Sigma_t} \Re\left(\left(\Box^{\mathbb{C}}_g\phi - \Box^{\mathbb{C}}_{m^*}\phi\right)\frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*}w\right)\,dx\,dt\right| + \tag{4.36}$$

$$\left|\int_{[0,T]\times\Sigma_t\setminus\{C_{max}\}} \Re\left(\left(\Box^{\mathbb{C}}_g\phi - \Box^{\mathbb{C}}_{m^*}\phi\right)\frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*}w\right)\,dx\,dt\right| \lesssim \epsilon E[\phi](1 + \|F\|_{L^\infty[w]}) + C_{err}[\phi]. \tag{4.37}$$

Proof. We now consider the reduced derivative

$$\tilde{D}_\alpha = \partial_\alpha + iA_\alpha.$$

Under the harmonic gauge, we have

$$\Box^{\mathbb{C}}_g\phi - \Box^{\mathbb{C}}_{m^*}\phi = \left(H^{\alpha\beta} + (\tilde{m}^{-1} - m^{*-1})^{\alpha\beta}\right)\tilde{D}_\alpha\tilde{D}_\beta\phi + (m^{*\alpha\beta}\tilde{D}_\alpha\tilde{D}_\beta\phi - \Box^{\mathbb{C}}_{m^*}\phi). \tag{4.38}$$

Defining

$$\widetilde{H}^{\alpha\beta} = H^{\alpha\beta} + (\tilde{m}^{-1} - m^{*-1})^{\alpha\beta}, \tag{4.39}$$

it is clear that $\widetilde{H}$ and $\partial\widetilde{H}$ satisfy the same L^∞ bounds in equation (2.14) as the corresponding quantities for H.

Given this, we have

$$m^{*\alpha\beta}\tilde{D}_\alpha\tilde{D}_\beta\phi - \Box^{\mathbb{C}}_{m^*}\phi = -\frac{1}{\sqrt{|m^*|}}\partial_\alpha(m^{*\alpha\beta}\sqrt{|m^*|})D_\beta\phi. \tag{4.40}$$

Thus, we have the bound

$$\left|m^{*\alpha\beta}\tilde{D}_\alpha\tilde{D}_\beta\phi - \Box^{\mathbb{C}}_{m^*}\phi\right| \lesssim M\tau_+^{-2}|D\phi|. \tag{4.41}$$

Therefore, using Hölder's inequality and the inequality $s + 2\iota < 1$,

$$\left\| \left(m^{*\alpha\beta} \tilde{D}_\alpha \tilde{D}_\beta \phi - \Box^{\mathbb{C}}_{m^*} \phi \right) \frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*} w \right\|_1 \lesssim M \left\| \tau_+^{-1/2-\iota} D\phi w^{1/2} \right\|_2 \left\| \tau_+^{-s} \tau_-^{-1/2-\iota} \frac{D_{\overline{K_0^s}}(r^*\phi)}{r^*} w^{1/2} \right\|_2 . \quad (4.42)$$

Therefore, by the inequalities $M < \epsilon$, and (4.11a) and (4.11f), we have

$$\left\| \left(m^{*\alpha\beta} \tilde{D}_\alpha \tilde{D}_\beta \phi - \Box^{\mathbb{C}}_{m^*} \phi \right) \frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*} w \right\|_1 \lesssim \epsilon E[\phi] \quad (4.43)$$

Next we want to bound the quantity

$$\int_0^T \int_{\Sigma_t} \widetilde{H}^{\alpha\beta} \tilde{D}_\alpha \tilde{D}_\beta \phi \frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*} w \, dx \, dt \quad (4.44)$$

We define the remainder momentum density tensor

$$\tilde{Q}[\phi]^\beta_{\overline{K_0^s}} = \Re \left(\frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*} \widetilde{H}^{\beta\gamma} D_\gamma \phi - \frac{1}{2} \overline{K_0^s}^\beta \widetilde{H}^{\gamma\delta} D_\gamma \phi \overline{D_\delta \phi} \right). \quad (4.45)$$

and apply the divergence theorem to this quantity in Euclidean space, in which we can ignore the distinction between D and $\tilde{D}$. As usual, we conduct our analysis in the extended exterior $r > \frac{t}{2}$, as we can use uniform estimates for the interior. We have by the divergence theorem

$$\int_0^T \int_{\Sigma_t} \partial_\beta \left(\tilde{Q}[\phi]^\beta_{\overline{K_0^s}} \tilde{w} \right) dx \, dt = \int_{\Sigma_T} \tilde{Q}[\phi]^0_{\overline{K_0^s}} \tilde{w} \, dx - \int_{\Sigma_0} \tilde{Q}[\phi]^0_{\overline{K_0^s}} . \tilde{w} \, dx \quad (4.46)$$

Our goal here is to isolate the desired quantity, (4.44), on the left hand side and move the rest, including the time-slice integrals, over to the right hand side, where we bound these quantities using the energy.

When the derivative falls on ϕ or $D\phi$, we get

$$\Re \left(\frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*} \widetilde{H}^{\beta\gamma} D_\beta D_\gamma \phi \right) + \frac{\overline{K_0^s}(r^*)}{r^*} \widetilde{H}^{\beta\gamma} D_\gamma \phi \overline{D_\beta \phi} + \Re \left(F_{\beta \overline{K_0^s}} \phi H^{\beta\gamma} \overline{D_\gamma \phi} \right).$$

The first term is precisely what we are looking for, and the second term can be bounded in the L^1 spacetime norm using (4.13). We now look at the third term. We note that

$$|F_{\underline{L}^*\overline{K_0^s}}| \lesssim \tau_+^{-1+s}\tau_-^{-1+s}\left\|F\right\|_{L^\infty[w]}, \tag{4.47a}$$

$$|F_{S_j^*\overline{K_0^s}}| \lesssim \tau_+^{-3/2+s}\tau_-^{-1/2+s}\left\|F\right\|_{L^\infty[w]}, \tag{4.47b}$$

$$|F_{L^*\overline{K_0^s}}| \lesssim \tau_+^{-1-s}\tau_-^{-1+3s}\left\|F\right\|_{L^\infty[w]}. \tag{4.47c}$$

Expanding in the null frame and combining this with the bounds

$$\epsilon\left\|\tau_-^{-1/2+s}\tau_+^{-3/2+s-1+\iota}\phi D\phi w\right\|_1 \lesssim \epsilon\left\|\tau_+^{-1}\phi w^{1/2}\right\|_2\left\|\tau_+^{-1/2-\iota}\tau_-^{s} D\phi w^{1/2}\right\|_2, \tag{4.48a}$$

$$\epsilon\left\|\tau_-^{-1+s}\tau_+^{-1+s-1+\iota}\phi\overline{D}\phi w\right\|_1 \lesssim \epsilon\left\|\tau_+^{-1}\phi w^{1/2}\right\|_2\left\|\tau_+^{s-1+\iota}\overline{D}\phi w^{1/2}\right\|_2, \tag{4.48b}$$

$$\epsilon\left\|\tau_-^{-1+s+\gamma'}\tau_+^{-1+s-1-\gamma'+\iota}\phi D\phi w\right\|_1 \lesssim \epsilon\left\|\tau_+^{-1}\phi w^{1/2}\right\|_2\left\|\tau_+^{-1-\iota}\tau_-^{s} D\phi w^{1/2}\right\|_2, \tag{4.48c}$$

gives us the uniform bound

$$\left|\Re\left(F_{\beta\overline{K_0^s}}\phi H^{\beta\gamma}\overline{D_\gamma\phi}\right)\right| \lesssim \epsilon E[\phi](\left\|F\right\|_{L^\infty[w]}).$$

Now we look at when the derivative falls on metric terms of (4.45). We have the estimates

$$\left|(\partial_\beta\widetilde{H}^{\beta\gamma})D_\gamma\phi\right| \lesssim \epsilon\left(\tau_+^{-1-\gamma'+\iota}\tau_-^{\gamma'-1}|D\phi| + \tau_+^{-1+\iota}\tau_-^{-1}|\overline{D}\phi|\right) \tag{4.49a}$$

$$\left|\overline{K_0^s}(\widetilde{H}^{\gamma\delta})D_\gamma\phi\overline{D_\delta\phi}\right| \lesssim \epsilon\left(\tau_+^{2s-2-\gamma'+\iota}\tau_-^{\gamma'}|D\phi|^2 + \tau_+^{2s-2+\iota}|D\phi||\overline{D}\phi|\right) \tag{4.49b}$$

from which follow

$$\left\|\frac{\overline{D_{\overline{K_0^s}}(r^*\phi)}}{r^*}(\partial_\beta\widetilde{H}^{\beta\gamma})D_\gamma\phi w\right\|_1 \lesssim \epsilon\left\|\tau_+^{-s}\tau_-^{-1/2-\iota}\frac{D_{\overline{K_0^s}}(r^*\phi)}{r^*}w^{1/2}\right\|_2\left(\left\|\tau_+^{-1-\iota}\tau_-^{1/2}|D\phi|w^{1/2}\right\|_2 + \left\|\tau_+^{s-1/2-\iota}|\overline{D}\phi|w^{1/2}\right\|_2\right) \tag{4.50a}$$

$$\left\|\overline{K_0^s}(H^{\gamma\delta})D_\gamma\phi\overline{D_\delta\phi}w\right\|_1 \lesssim \epsilon\left(\left\|\tau_+^{-1/2-\iota}\tau_-^{s}|D\phi|w^{1/2}\right\|_2^2 + \left\|\tau_+^{2s-1/2-\iota}|\overline{D}\phi|\right\|_2\left\|\tau_+^{-1/2-\iota}|D\phi|w^{1/2}\right\|_2\right) \tag{4.50b}$$

where the last estimate comes from Hölder's inequality and the condition $s-4\iota > 1/2$. The inequality follows.

Next, we consider the case where the derivative falls on $\overline{K_0^s}$. This is fortunately more straightforward, since for most terms which appear we do not need to differentiate derivatives. We have the terms

$$\Re\left(\widetilde{H}^{\beta\gamma}\partial_\beta\left(\frac{\overline{K_0^s}(r^*)}{r^*}\right)\overline{\phi}D_\gamma\phi - \frac{1}{2}(\partial_\beta\overline{K_0^s}^\beta)H^{\gamma\delta}D_\gamma\phi\overline{D_\delta\phi} + (\partial_\beta\overline{K_0^s}^\alpha)\overline{D_\alpha\phi}\widetilde{H}^{\alpha\beta}D_\beta\phi\right). \tag{4.51}$$

We can bound the first two terms on the right by

$$\left\|\widetilde{H}^{\beta\gamma}\partial_\beta\left(\frac{\overline{K_0^s}(r^*)}{r^*}\right)\overline{\phi}D_\gamma\phi w\right\|_1 \lesssim \epsilon\left\|\tau_+^{2s-3+\iota}|\phi||D\phi|w\right\|_1, \tag{4.52a}$$

$$\lesssim \epsilon\left\|\tau_+^{s-3/2-\iota}|\phi|w^{1/2}\right\|_2\left\|\tau_+^{s-3/2+2\iota}|D\phi|w^{1/2}\right\|_2,$$

$$\left\|(\partial_\beta\overline{K_0^s}^\beta)H^{\gamma\delta}D_\gamma\phi\overline{D_\delta\phi}w\right\|_1 \lesssim \epsilon\left(\left\|\tau_+^{2s-2+\iota}|D\phi||\overline{D}\phi|w\right\|_1+\left\|\tau_+^{2s-2-\gamma'+\iota}\tau_-^{\gamma'}|D\phi|^2w\right\|_1\right). \tag{4.52b}$$

Both of these fit in our energy norm as a direct consequence of (4.11). For the third term, in the exterior we have the terms

$$|B_j(\overline{K_0^s}^\alpha)D_\alpha\phi| \lesssim |\tau_+^{2s-1}D_{B_j}\phi|, \tag{4.53a}$$

$$|L^*(\overline{K_0^s}^\alpha)D_\alpha\phi| \lesssim |\tau_+^{2s-1}D_{L^*}\phi|+|\tau_-^{2s}\tau_+^{-1}D_{\underline{L}^*}\phi|, \tag{4.53b}$$

$$|\underline{L}^*(\overline{K_0^s}^\alpha)D_\alpha\phi| \lesssim \tau_+^{2s-1}|D\phi|. \tag{4.53c}$$

Thus,

$$\left|\partial_\beta\overline{K_0^s}^\alpha)\overline{D_\alpha\phi}\widetilde{H}^{\alpha\beta}D_\beta\phi\right| \lesssim \epsilon\left(\tau_+^{2s-2+\iota}|\overline{D}\phi||D\phi|+\tau_-^{2s}\tau_+^{-2+\iota}|D\phi|^2+\tau_-^{\gamma'}\tau_+^{2s-2-\gamma'+\iota}|D\phi|^2\right). \tag{4.54}$$

The L^1 norms corresponding to these terms are therefore contained in our energy norm.

Finally, we look at the case when the derivative falls on $\tilde{w}$. We don't have any nice sign condition, so we use the pointwise estimates

$$\left\|\tilde{Q}[\phi]^\beta_{\overline{K_0^s}}\partial_\beta w\right\| \lesssim \epsilon\left\|\tau_+^{-s}\tau_-^{-1/2-\iota}\frac{D_{\overline{K_0^s}}(r^*\phi)}{r^*}w^{1/2}\right\|_2\left(\left\|\tau_+^{s-1-\gamma'+\iota}\tau_-^{1/2+\iota}|D\phi|w^{1/2}\right\|_2+\left\|\tau_+^{s-1+\iota}\tau_-^{1/2+\iota-1}|\overline{D}\phi|w^{1/2}\right\|_2\right)+$$

$$+\epsilon\left\|\tau_+^{2s-2+\iota}|D\phi||\overline{D}\phi|w\right\|_1+\epsilon\left\|\tau_+^{2s-2-\gamma'+\iota}\tau_-^{\gamma'}|D\phi|^2w\right\|_1. \tag{4.55}$$

By the norms in (4.11), the right hand side can be bounded by $\epsilon E[\phi](T)$. ☐

5 L^∞ Estimates: F

We now establish L^∞ estimates on the charge-modified field $\widetilde{F}$. This is for the most part similar to calculations in [16], up to our modified frame, along with some additional computations in the end to establish our pure L^∞ estimate on nice components. In particular, our primary tools are Lemma 9.6 using equation (2.8) combined with (2.5) to commute the Lorentz fields through our frame. First, we take an estimate which holds for bad components in the extended exterior.

Lemma 5.1. *We have the following uniform estimate on all components of* $\widetilde{F}$*:*

$$\tau_+\tau_-^{s+1/2}|\chi\widetilde{F}|w^{1/2} \lesssim E_2^{1/2}[\widetilde{F}] \tag{5.1}$$

Proof. We look at the extended exterior, $r^* > t/2$. This follows from (9.15) with $\delta_+ = 0$ and $\delta_- = s$, using the inequality

$$|\tau_-\partial_{r^*}\phi| \lesssim |S^*\phi| + |\Omega^*_{0r}\phi| + |\partial_{r^*}\phi|,$$

then expanding each of these with respect to the Lorentz fields. We have in particular the estimate

$$\left\|\tau_+\tau_-^{1/2+s}\chi|\widetilde{F}(\partial_{x^{*\alpha}},\partial_{x^{*\beta}})|w^{1/2}\right\|_{L^\infty(\mathbb{R}^3)} \lesssim \sum_{\substack{|I|\leq 2\\ X\in\mathbb{L}}}\left\|\tau_-^s X^I(\chi\widetilde{F}(\partial_{x^{*\alpha}},\partial_{x^{*\beta}}))w^{1/2}\right\|_{L^2(\mathbb{R}^3)}. \tag{5.2}$$

Since we have the estimate $\partial^\alpha\chi \lesssim \tau_+^{-|\alpha|}$, we can ignore it in our calculations. We use the commutator estimates (2.4) combined with repeated application of the identity (2.8) which gives our result.

$\square$

Lemma 5.2. *For all components of* $\widetilde{F}$*, we have the following estimate:*

$$\tau_+^{s+3/2}|(1-\chi)\widetilde{F}|w^{1/2} \lesssim E_2^{1/2}[\widetilde{F}]. \tag{5.3}$$

The proof for this is virtually identical, using the estimate (9.17) and noting that $\tau_- \approx \tau_+$ in the support of $(1-\chi)$. We can use this to extend the result of Lemma 5.1 to the far interior (i.e., remove the χ from the left hand side). Now we consider the behavior of the nicer components.

Lemma 5.3. *For the nice components* α, ρ, σ *of* $\widetilde{F}$ *we have the estimate*

$$\tau_+^{s+1}\tau_-^{1/2}|\widetilde{\not{F}}|w^{1/2} \lesssim E_2^{1/2}[\widetilde{F}]. \tag{5.4}$$

Proof. Here we need to look at Lemma 2.2. For any component σ, ρ, α, we first have the Sobolev estimate

$$\tau_+^{s+1}\tau_-^{1/2}|\widetilde{\not{F}}|w^{1/2} \lesssim \sum_{\substack{|I|\leq 2\\ X\in\mathbb{L}}}\left\|\tau_+^s X^I(\chi\widetilde{\not{F}})w^{1/2}\right\|_{L^2(\mathbb{R}^3)}. \tag{5.5}$$

We can ignore the χ term on the right hand side. Repeated application of Lemma 2.2 gives us our desired estimate. $\square$

We now look at the $L^2(L^\infty)$ estimate for α. We use a slightly simpler method than in [16] here, using Lemma 2.2. In particular, we do not need to exploit the radial boost field. We state this as following:

Lemma 5.4. *For α, we have*

$$\left\|\tau_+^{s+1}\tau_-^{1/2}\alpha[\widetilde{F}](w')^{1/2}\right\|_{L^2(t)L^\infty(x)} \lesssim S_2[\widetilde{F}](T). \tag{5.6}$$

Proof. We as usual take our Sobolev estimate in the extended exterior

$$\left\|\tau_+^{s+1}\tau_-^{1/2}\chi\alpha[\widetilde{F}](w')^{1/2}\right\|^2_{L^\infty(x)} \lesssim \sum_{\substack{|I|\leq 2\\ X\in\mathbb{L}}} \left\|\tau_+^{s}X^I(\chi\alpha[\widetilde{F}])(w')^{1/2}\right\|^2_{L^2(x)}. \tag{5.7}$$

We can use the usual expansion in terms of Lie derivatives, followed by the commutator estimate 2.2, and it follows that this is contained in $S_2[F](T)$. We combine this with the interior estimate

$$\left\|\tau_+^{3/2+s}|\widetilde{F}_{\alpha\beta}|(w')^{1/2}\right\|_{L^\infty(x)} \lesssim \sum_{X\in\mathbb{L}.|I|\leq 2} \left\|\tau_+^{s}X^I((1-\chi)\widetilde{F}_{\alpha\beta})(w')^{1/2}\right\|_{L^2(x)} \tag{5.8}$$

coming from (9.17), to get the full inequality. $\square$

We can combine these estimates as follows:

Lemma 5.5. *For any two-form $\widetilde{F}$ with zero charge and sufficient decay, the following estimate holds:*

$$\tau_+\tau_-^{s+1/2}|\underline{\alpha}[\widetilde{F}]| + \tau_+^{s+1}\tau_-^{1/2}(|\alpha[\widetilde{F}]| + |\rho[\widetilde{F}]| + |\sigma[\widetilde{F}]|) + \left\|\tau_+^{s+1}\tau_-^{1/2}\alpha[\widetilde{F}](w')^{1/2}\right\|_{L^2(t)L^\infty(x)} \lesssim \mathcal{E}_2[\widetilde{F}](T). \tag{5.9}$$

In general we use this estimate on $\widetilde{F}$ and Lie derivatives of this quantity.

Before we proceed, we mention one auxiliary estimate which will give us more precise bounds for the component α. We recall the conical energy

$$C_0[F](T) = \sup_{u^*}\int_{\{C(u^*)\}\cap\{t\in[0,T]\}} \left(\frac{1}{2}\tau_+^{2s}Q_{\tilde{L^*}L^*} + \frac{1}{2}\tau_-^{2s}Q_{\tilde{L^*}\underline{L}^*}\right) w\, dVC(u^*).$$

We have the formula

$$(\tilde{L^*}-L^*)^{L^*}\lesssim H^{L^*\underline{L}^*}\qquad(\tilde{L^*}-L^*)^{\underline{L}^*}\lesssim g^{\underline{L}^*\underline{L}^*}\qquad\qquad(\tilde{L^*}-L^*)^{S^*_j}\lesssim g^{\underline{L}^*S^*_j}.\tag{5.10}$$

We can combine this with the estimates (3.30) to get the bounds

$$\left|Q_{\tilde{L^*}L^*}-|\alpha|^2\right|\lesssim\epsilon|\alpha|^2+\epsilon\tau_+^{-1-\gamma'+\iota}\tau_-^{\gamma'}|\not{F}|^2+\epsilon^2\tau_-^{2\gamma}\tau_+^{-2-2\gamma+2\iota}|F|^2,\tag{5.11a}$$

$$\left|Q_{L^*\underline{L}^*}-(|\rho|^2+|\sigma|^2)\right|\lesssim\epsilon|\not{F}|^2+\epsilon\tau_-^{\gamma'}\tau_+^{-1-\gamma'+\iota}|F|^2.\tag{5.11b}$$

Therefore, it follows that for sufficiently small ϵ, and for some C independent of ϵ,

$$\tau_+^{2s}Q_{\tilde{L^*}L^*}+\frac{1}{2}\tau_-^{2s}Q_{\tilde{L^*}\underline{L}^*}\geq 1/2(\tau_+^{2s}|\alpha|^2+\tau_-^{2s}(|\rho|^2+|\sigma|^2))-C\epsilon\tau_-^{2s+\gamma'}\tau_+^{-1-\gamma'+\iota}|F|^2.\tag{5.12}$$

The last term is not a part of our conical energy. However, we can bound the last term by

$$\epsilon\tau_-^{2s+\gamma'}\tau_+^{-1-\gamma'+\iota}|G|^2\lesssim\epsilon\tau_-^{\gamma'-1}\tau_+^{-3-\gamma'+\iota}\mathcal{E}_2[G](T).\tag{5.13}$$

It follows that

$$\sup_{u^*}\int_{\{C(u^*)\}\cap\{t\in[0,T]\}}\epsilon\tau_-^{2s+\gamma'}\tau_+^{-1-\gamma'+\iota}|G|^2\lesssim\epsilon\mathcal{E}_2[G](T).\tag{5.14}$$

The estimate (3.50) follows.

Letting G be any of the field quantities appearing in $\mathcal{E}_2[F](T)$ and applying the equation (9.16) gives us the following:

$$\left\|\tau_+^{3/2+s}\alpha[\widetilde{F}]w\right\|_{L^\infty(C_{u^*})}\lesssim\sum_{\substack{|I|,|J|\leq 1\\ X\in\{\underline{u}^*L^*,\mathbb{O}\},Y\in\mathbb{O}}}\left\|\tau_+^{2s}\left(\alpha[\mathcal{L}_X^I\mathcal{L}_Y^J\widetilde{F}]+\tau_+^{2s}\tau_0\left(\rho[\mathcal{L}_X^I\mathcal{L}_Y^J\widetilde{F}]+\sigma[\mathcal{L}_X^I\mathcal{L}_Y^J\widetilde{F}]\right)\right)w\right\|_{L^2(C_{u^*})}.\tag{5.15}$$

Applying equation (5.12) gives us the estimate

$$|\tau_+^{3/2+s}\alpha[\widetilde{F}]w^{1/2}|\lesssim\mathcal{E}_4^{1/2}[\widetilde{F}](T).\tag{5.16}$$

We can combine these to get the following:

Theorem 5.6. *For any two-form $\widetilde{F}$ with zero associated charge and sufficient decay, the following estimates* *hold:*

$$\tau_+\tau_-^{s+1/2}|\underline{\alpha}[\widetilde{F}]| + \tau_+^{s+1}\tau_-^{1/2}(|\alpha[\widetilde{F}]| + |\rho[\widetilde{F}]| + |\sigma[\widetilde{F}]|) + \left\|\tau_+^{s+1}\tau_-^{1/2}\alpha[\widetilde{F}](w')^{1/2}\right\|_{L^2(t)L^\infty(x)} \lesssim \mathcal{E}_2^{1/2}[\widetilde{F}](T), \tag{5.17a}$$

$$|\tau_+^{3/2+s}\alpha[\widetilde{F}]w^{1/2}| \lesssim \mathcal{E}_4^{1/2}[\widetilde{F}](T). \tag{5.17b}$$

6 L^∞ Estimates: ϕ

We now establish analogous L^∞ estimates on ϕ. Our analysis once again revolves around theorem 9.6, with some more terms to consider. We start out with the bound

$$\left(\tau_+^2\tau_-^{2s+1}|D\phi|^2 + \tau_+^2\tau_-^{2s+1}\sum_{\substack{|I|\le 1\\ X\in\mathbb{L}}}\left|\frac{D_X\phi}{\tau_+}\right|^2\right) w \lesssim \sum_{\substack{|I|\le 2\\ X\in\mathbb{L}}}\left\|\tau_-^s D_X^I D\phi w^{1/2}\right\|_{L^2(x)}^2 + \sum_{\substack{|I|\le 3\\ X\in\mathbb{L}}}\left\|\tau_-^s\tau_+^{-1}D_X^I\phi w^{1/2}\right\|_{L^2(x)}^2. \tag{6.1}$$

The second term is almost immediately contained in $E_2[\phi](T)$, so we only need to deal with the first term. Im particular, we bound this by commuting D through D_X^I, as we have the bound

$$\sum_{\substack{|I|\le 2\\ X\in\mathbb{L}}}\left\|\tau_-^s D D_X^I\phi w^{1/2}\right\|_{L^2(x)} \lesssim (E_2[\phi](T))^{1/2}.$$

We look at the case $|I| = 2$, as other cases are easier. First, we have the identity

$$[D_{X_1}D_{X_2}, D_{x^{*\alpha}}]\phi = D_{X_1}(iF(X_2,\partial_{x^{*\alpha}})\phi) + D_{X_1}D_{[X_2,\partial_{x^{*\alpha}}]}\phi + iF(X_1,\partial_{x^{*\alpha}})D_{X_2}\phi + D_{[X_1,\partial_{x^{*\alpha}}]}D_{X_2}\phi. \tag{6.2}$$

We take two simplifications: first, we use the identity (2.8) to rewrite

$$D_{X_1}(iF(X_2,\partial_{x^{*\alpha}})\phi) = iF(X_2,\partial_{x^{*\alpha}})D_{X_1}\phi + i(\mathcal{L}_{X_1}F)(X_2,\partial_{x^{*\alpha}})\phi + iF([X_1,X_2],\partial_{x^{*\alpha}})\phi + iF(X_2,[X_1,\partial_{x^{*\alpha}}])\phi. \tag{6.3}$$

Using the estimate

$$\left\|\tau_+\tau_-F\right\|_{L^\infty(x)} \lesssim \|F\|_{L^\infty[w]}, \tag{6.4}$$

and our usual commmutator relations, we can write

$$|D_{X_1}(iF(X_2,\partial_{x^{*\alpha}})\phi)| \lesssim \tau_+ \sum_{\substack{|I|\leq 1\\ X\in\mathbb{L}}} |\mathcal{L}_X F| \sum_{\substack{|I|\leq 1\\ Y\in\mathbb{L}}} |D_Y\phi|, \tag{6.5}$$

and therefore

$$\sum_{X_i\in\mathbb{L}} \left\|\tau_-^s D_{X_1}(iF(X_2,\partial_{x^{*\alpha}})\phi)w^{1/2}\right\|_{L^2(x)} \lesssim \sum_{\substack{|I|\leq 1\\ X\in\mathbb{L}}} \|\mathcal{L}_X F\|_{L^\infty[w]} \sum_{\substack{|I|\leq 1\\ Y\in\mathbb{L}}} \left\|\tau_-^{s-1} D_Y\phi w^{1/2}\right\|_{L^2(x)}. \tag{6.6}$$

An application of Lemma 9.10 with $p = 2s-2$, $q = 0$ gives us

$$\left\|\tau_-^{s-1} D_Y\phi w^{1/2}\right\|_{L^2(x)} \lesssim (E_1[\phi](T))^{1/2}. \tag{6.7}$$

Now we look at $D_{X_1}D_{[X_2,\partial_{x^{*\alpha}}]}\phi$. We once again commute

$$D_{X_1}D_{[X_2,\partial_{x^{*\alpha}}]}\phi = D_{[X_2,\partial_{x^{*\alpha}}]}D_{X_1}\phi + iF(X_1,[X_2,\partial_{x^{*\alpha}}])\phi + D_{[X_1,[X_2,\partial_{x^{*\alpha}}]]}\phi. \tag{6.8}$$

The first and third terms on the right hand side can be bounded by $|DD_{X_1}\phi|+|D\phi|$ and bounded by $E_1[\phi](T)$ in the usual way, and the second term can be dealt with using the bound

$$|iF(X_1,[X_2,\partial_{x^{*\alpha}}])\phi| \lesssim \tau_+|F||\phi|, \tag{6.9}$$

after which we can treat it like (6.3). The last two terms on the right hand side of (6.2) can be dealt with in a similar way. Likewise, the terms where $|I| \leq 1$ are more straightforward.

We therefore have our first estimate:

Lemma 6.1. *For a function ϕ defined on $[0,T]\times\Sigma_t$ with suitable decay at spatial infinity,*

$$\tau_+\tau_-^{s+1/2}\left(|D\phi| + \sum_{\substack{|I|\leq 1\\ X\in\mathbb{L}}}\left|\frac{D_X\phi}{\tau_+}\right|\right) w^{1/2} \lesssim \left(1+\sum_{\substack{|I|\leq 1\\ X\in\mathbb{L}}}\|\mathcal{L}_X F\|_{L^\infty[w]}\right)(E_2[\phi](T))^{1/2}. \tag{6.10}$$

The estimate on $|D\phi|$, of course, mainly serves to bound the worst derivative in our null decomposition, $D_{\underline{L}^*}\phi$. The estimate with our Lorentz fields comes up in our commutators in a natural way. We now take a look at our better derivatives. First, we consider angular derivatives D_{S^*}. Note that here we only care about the region $r^* \geq t/2$, as the interior case has nominally better estimates coming from the relation $\tau_- \approx \tau_+$.

As usual, we start off with the identity

$$\left\|\tau_+^{2+2s}\tau_-^1 |D_{S_i^*}\phi|^2 w\right\|_{L^\infty(x)} \lesssim \sum_{\substack{|I|\leq 2\\ X\in\mathbb{L}}} \left\|\tau_+^s D_X^I D_{S_i^*}\phi w^{1/2}\right\|_{L^2(x)}^2 . \tag{6.11}$$

It is clear that, in order for the right hand side to be bounded, we cannot have a derivative like $D_{\underline{L}^*}\phi$ come out (or an equivalent term like $\underline{\alpha}$) without some additional decay appearing.

We again can absorb the terms where $D_{S_i^*}$ commutes through into our energy, and look at the commutator. We take an analogous identity to (6.2):

$$[D_{X_1}D_{X_2}, D_{S_i^*}]\phi = D_{X_1}(iF(X_2, S_i^*)\phi) + D_{X_1}D_{[X_2,S_i^*]}\phi + iF(X_1, S_i^*)D_{X_2}\phi + D_{[X_1,D_{S_i^*}]}D_{X_2}\phi, \tag{6.12}$$

and

$$D_{X_1}(iF(X_2, D_{S_i^*})\phi) = iF(X_2, S_i^*)D_{X_1}\phi + i(\mathcal{L}_{X_1}F)(X_2, S_i^*)\phi + iF([X_1, X_2], S_i^*)\phi + iF(X_2, [X_1, S_i^*])\phi. \tag{6.13}$$

Between the commutator identities (2.4) and (2.5), we can bound this in magnitude by

$$|D_{X_1}(iF(X_2, D_{S_i^*})\phi)| \lesssim \sum_{\substack{|I|\leq 1\\ X\in\mathbb{L}}} \big(\tau_+(|\alpha(\mathcal{L}_X F)| + |\rho(\mathcal{L}_X F)| + |\sigma(\mathcal{L}_X F))| + \tau_-|\underline{\alpha}(\mathcal{L}_X F)|\big) \sum_{\substack{|I|\leq 1\\ Y\in\mathbb{L}}} |D_Y\phi| \tag{6.14}$$

We have the inequality

$$\tau_+^s\tau_-^{1-s}\left(\tau_+(|\alpha(\mathcal{L}_X F)| + |\rho(\mathcal{L}_X F)| + |\sigma(\mathcal{L}_X F))| + \tau_-|\underline{\alpha}(\mathcal{L}_X F)|\right) \lesssim \|\mathcal{L}_X F\|_{L^\infty(x)} . \tag{6.15}$$

It follows that

$$\sum_{\substack{|I|\leq 2\\ X\in\mathbb{L}}} \left\|\tau_+^s D_{X_1}(iF(X_2, D_{S_i^*})\phi w^{1/2}\right\|_{L^2(x)}^2 \lesssim \left(1 + \sum_{\substack{|I|\leq 1\\ X\in\mathbb{L}}} \|\mathcal{L}_X F\|_{L^\infty[w]}^2\right) \sum_{\substack{|I|\leq 1\\ Y\in\mathbb{L}}} \left\|\tau_-^{s-1} D_Y\phi w^{1/2}\right\|^2 . \tag{6.16}$$

We deal with all other terms in the same manner as the corresponding terms in (6.2), which, along with Lemma 9.10, gives us our second estimate:

Lemma 6.2. *For a function ϕ defined on $[0,T]\times\Sigma_t$ with suitable decay at spatial infinity,*

$$\tau_+^{s+1}\tau_-^{1/2}|D_{S_i^*}\phi|w^{1/2} \lesssim \left(1+\sum_{\substack{|I|\le 1\\ X\in\mathbb{L}}}\|\mathcal{L}_X F\|_{L^\infty[w]}\right)(E_2[\phi](T))^{1/2}. \tag{6.17}$$

We can now take a nicer estimate on ϕ. Our time-slice Sobolev estimate, (9.18), gives us

$$\tau_+\tau_-^{s-1/2}|\phi|w^{1/2} \lesssim \sum_{\substack{|I|\le 1\\ X\in\mathbb{L}}}\left\|\tau_-^{s-1}D_X^I\phi w^{1/2}\right\|_{L^2(x)}. \tag{6.18}$$

Again, we apply Lemma 9.10 to get the identity

Lemma 6.3. *For a function ϕ defined on $[0,T]\times\Sigma_t$ with suitable decay at spatial infinity,*

$$\tau_+\tau_-^{s-1/2}|\phi|w^{1/2} \lesssim E_2^{1/2}[\phi](T). \tag{6.19}$$

We now take an $L^2(t)L^\infty(x)$ estimate for the nice component $r^{*-1}D_{L^*}(r^*\phi)$ in the extended exterior region $r^*>t/2$. square our basic Sobolev estimate and integrate in time to get

$$\left\|\tau_+^{s+1}\tau_-^{1/2}\frac{D_{L^*}(r^*\phi)}{r^*}(w')^{1/2}\right\|_{L^2(t)L^\infty(x)} \lesssim \sum_{\substack{|I|,|J|\le 1\\ X\in\{\partial_{r^*},\Omega^*_{0r^*},Z^*\}\cup\mathbb{O},Y\in\mathbb{O}}}\left\|\tau_+^s D_X D_Y\left(\frac{D_{L^*}(r^*\phi)}{r^*}(w')^{1/2}\right)\right\|_2. \tag{6.20}$$

Here we have established greater restrictions on X,Y. Note that we have replaced $\tau_-\partial_{r^*}$ with $\partial_{r^*},\Omega^*_{0r^*},Z^*$, as

$$|\tau_-\partial_{r^*}\phi| \lesssim |\partial_{r^*}\phi|+|\Omega^*_{0r^*}\phi|+|Z^*\phi|.$$

We start with the commutator relation

$$\left[D_X,\frac{1}{r^*}D_{L^*}(r^*\cdot)\right]\phi = D_{[X,L^*]}\phi + iF(X,L^*)\phi + X\left(\frac{1}{r^*}\right)\phi \tag{6.21}$$

from which it follows that

$$\begin{aligned}\left[D_{X_1}D_{X_2},\frac{1}{r^*}D_{L^*}(r^*\cdot)\right]\phi = D_{X_1}\left(D_{[X_2,L^*]}\phi + iF(X_2,L^*)\phi + X_2\left(\frac{1}{r^*}\right)\phi\right) + \\ + D_{[X_1,L^*]}D_{X_2}\phi + i(F(X_1,L^*)D_{X_2}\phi) + X_1\left(\frac{1}{r^*}\right)D_{X_2}\phi.\end{aligned} \tag{6.22}$$

We must take our restricted set of vector fields $X_2 \in \{\Omega^*_{ij}\}$, $X_1 \in \{\partial_{r^*}, Z^*, \Omega^*_{0r^*}, \Omega^*_{ij}\}$. Recalling the identities (2.5), we have that this can be simplified to

$$\left[D_{\partial_{r^*}} D_{\Omega^*_{ij}}, \frac{1}{r^*} D_{L^*}(r^* \cdot)\right] \phi = i D_{\partial_{r^*}} (F(\Omega^*_{ij}, L^*)\phi) + iF(\partial_{r^*}, L^*) D_{\Omega^*_{ij}} \phi - \frac{1}{r^{*2}} D_{\Omega^*_{ij}} \phi. \tag{6.23}$$

We lok at the first term on the right, which we can first simplify to

$$D_{\partial_{r^*}} (F(\Omega^*_{ij}, L^*)\phi) = (\mathcal{L}_{\partial_{r^*}} F)(\Omega^*_{ij}, L^*)\phi + F(\Omega^*_{ij}, L^*) D_{\partial_{r^*}} \phi. \tag{6.24}$$

Using the identity

$$\mathcal{L}_{fX} F(Y, Z) = f \cdot (\mathcal{L}_X F)(Y, Z) + (Yf)F(X, Z) + (Zf)F(Y, X), \tag{6.25}$$

where f is a function, and decomposing ∂_{r^*}, we have

$$(\mathcal{L}_{\partial_{r^*}} F)(\Omega^*_{ij}, L^*) = \omega^k (\mathcal{L}_{\partial_{x^{*k}}} F)(\Omega^*_{ij}, L^*) + \frac{1}{r^*} F(\Omega^*_{ij}, L^*). \tag{6.26}$$

We next have

$$\left[D_{\Omega^*_{0r^*}} D_{\Omega^*_{ij}}, \frac{1}{r^*} D_{L^*}(r^* \cdot)\right] \phi = D_{\Omega^*_{0r^*}} \left(iF(\Omega^*_{ij}, L^*)\phi\right) - \frac{D_{L^*}(r^* D_{\Omega^*_{ij}} \phi)}{r^*} + iF(\Omega^*_{0r^*}, L^*) D_{\Omega^*_{ij}} \phi + \frac{r^* - t}{r^{*2}} D_{\Omega^*_{ij}} \phi. \tag{6.27}$$

We can again rewrite the first term as

$$D_{\Omega^*_{0r^*}} \left(iF(\Omega^*_{ij}, L^*)\phi\right) = i(\mathcal{L}_{\Omega^*_{0r^*}} F)(\Omega^*_{ij}, L^*)\phi - iF(\Omega^*_{ij}, L^*)\phi + iF(\Omega^*_{ij}, L^*) D_{\Omega^*_{0r^*}} \phi \tag{6.28}$$

and decompose further

$$(\mathcal{L}_{\Omega^*_{0r^*}} F)(\Omega^*_{ij}, L^*) = \omega^k (\mathcal{L}_{\Omega^*_{0k}} F)(\Omega^*_{ij}, L^*) + \frac{t}{r^*} F(\Omega^*_{ij}, L^*). \tag{6.29}$$

Next, we have

$$\left[D_{\Omega^*_{kl}} D_{\Omega^*_{ij}}, \frac{1}{r^*} D_{L^*}(r^* \cdot)\right] \phi = i D_{\Omega^*_{kl}} (F(\Omega^*_{ij}, L^*)\phi) + iF(\Omega^*_{kl}, L^*) D_{\Omega^*_{ij}} \phi. \tag{6.30}$$

Once again, we can rewrite the first term as

$$D_{\Omega^*_{kl}}(iF(\Omega^*_{ij},L^*)\phi) = i(\mathcal{L}_{\Omega^*_{kl}}F)(\Omega^*_{ij},L^*) + iF([\Omega^*_{kl},\Omega^*_{ij}],L^*)\phi + iF(\Omega^*_{ij},L^*)D_{\Omega^*_{kl}}\phi. \tag{6.31}$$

Finally, we look at the scaling field Z^*. As usual, we have

$$\left[D_{Z^*}D_{\Omega^*_{ij}}, \frac{1}{r^*}D_{L^*}(r^*\cdot)\right]\phi = D_{Z^*}(iF(\Omega^*_{ij},L^*)\phi) - \frac{D_{L^*}(r^*D_{\Omega^*_{ij}}\phi)}{r^*} + iF(Z^*,L^*)D_{\Omega^*_{ij}}\phi \tag{6.32}$$

along with

$$D_{Z^*}(iF(\Omega^*_{ij},L^*)\phi) = i(\mathcal{L}_{Z^*}F)(\Omega^*_{ij},L^*)\phi - iF(\Omega^*_{ij},L^*)\phi + iF(\Omega^*_{ij},L^*)D_{Z^*}\phi. \tag{6.33}$$

Combining the identities appearing in (6.23)-(6.33) gives us the exterior estimate

$$\begin{aligned}\sum_{\substack{|I|,|J|\le 1\\ X\in\{\partial_{r^*},\Omega^*_{0r^*},Z^*\}\cup\mathbb{O},Y\in\mathbb{O}}} \left|D_XD_Y\left(\frac{D_{L^*}(r^*\phi)}{r^*}\right)\right| &\lesssim \sum_{\substack{|I|\le 2\\ X\in\mathbb{L}}}\left|\left(\frac{D_{L^*}(r^*D^I_X\phi)}{r^*}\right)\right| + \sum_{\substack{|I|\le 1\\ X\in\mathbb{L}}}\tau_0\left|\frac{D^I_X\phi}{r^*}\right| + \\ &+ \sum_{\substack{|I|+|J|\le 2\\ X,Y\in\mathbb{L}}}\tau_+|\alpha(\mathcal{L}_XF)||D_Y\phi| + \sum_{\substack{|I|+|J|\le 2\\ X,Y\in\mathbb{L}}}\tau_-|\rho(\mathcal{L}_XF)||D_Y\phi|.\end{aligned} \tag{6.34}$$

It suffices to show that all terms appearing on the right hand side of (6.34) can be bounded by the energy when inserted into (6.20). Due to the energy considerations coming from the metric, we focus on bounding them by $\mathcal{E}_5[\phi](T)$, rather than $\mathcal{E}_2$, which makes calculations significantly easier. The first term on the right appears in $\mathcal{E}_2[\phi](T)$.

The term with $\tau_+^s\tau_0|D^I_X\phi/r(w')^{1/2}|$ is similarly easy to bound, as it appears in $\mathcal{E}_3$ using the bound $1/2+\iota\le 1$.

In order to bound $\tau_+|\alpha(\mathcal{L}_XF)||D_Y\phi|$ we take the following:

$$\begin{aligned}\left\|\tau_+^{s+1}|\alpha(\mathcal{L}_X\widetilde{F})||D_Y\phi|(w')^{1/2}\right\|_2 &\lesssim |\tau_+D_Y\phi|\left\|\tau_+^s\alpha(\mathcal{L}_X\widetilde{F})(w')^{1/2}\right\|_2, \qquad (6.35a)\\ &\lesssim \mathcal{E}_3[\phi](T)\mathcal{E}_1[F](T),\\ \left\|\tau_+^{s+1}|\alpha(\mathcal{L}_X\overline{F})||D_Y\phi|(w')^{1/2}\right\|_2 &\lesssim q\left\|\tau_0\tau_+^s\left|\frac{D_Y\phi}{r^*}\right|(w')^{1/2}\right\|_2, \qquad (6.35b)\\ &\lesssim q\mathcal{E}_2[\phi](T).\end{aligned}$$

Finally, we look at the term $\tau_-|\rho(\mathcal{L}_X F)||D_Y\phi|$ Again we take the decomposition

$$\begin{aligned}
\left\|\tau_+^s\tau_-|\rho(\mathcal{L}_X\widetilde{F})||D_Y\phi|(w')^{1/2}\right\|_2 &\lesssim |\tau_+D_Y\phi|\left\|\tau_+^s\tau_0\rho(\mathcal{L}_X\widetilde{F})(w')^{1/2}\right\|_2 && (6.36a)\\
&\lesssim \mathcal{E}_3[\phi](T)\mathcal{E}_1[F](T),\\
\left\|\tau_+^s\tau_-|\rho(\mathcal{L}_X\overline{F})||D_Y\phi|(w')^{1/2}\right\|_2 &\lesssim q\left\|\tau_0\tau_+^s\left|\frac{D_Y\phi}{r^*}\right|(w')^{1/2}\right\|_2, && (6.36b)\\
&\lesssim q\mathcal{E}_2[\phi](T).
\end{aligned}$$

We therefore have the estimate

$$\left\|\tau_+^{s+1}\tau_-^{1/2}\frac{D_{L^*}(r^*\phi)}{r^*}(w')^{1/2}\right\|_{L^2(t)L^\infty(x)} \lesssim (1+|q|+\mathcal{E}_1[F](T))\mathcal{E}_3[\phi](T). \tag{6.37}$$

This is not an optimal number of derivatives. However, this is not of concern as the metric term dominates the number of derivatives needed.

We now consider the interior terms. Note that here we can expect all derivatives to satisfy similar decay, so we can estimate, by our interior time-slice estimate (9.20)

$$\left\|\tau_+^{s+1-\iota}|D\phi|\right\|^2_{L^2(t)L^\infty(x)\{r>t/2\}} \lesssim \sum_{\substack{|I|\le 2\\ X\in\mathbb{L}}}\left\|\tau_+^{s-1/2-\iota}D_X^ID\phi\right\|^2_{L^2(t)L^2(x)\{r<3t/4\}}.$$

We can commute D through D_X^I, noting that commutators look like DD_Y, where Y is one of our Lorentz fields, and, noting that $\tau_+\approx\tau_-$ and $\tau_0\approx 1$, we can conclude

$$\left\|\tau_+^{s+1-\iota}|D\phi|\right\|_{L^2(t)L^\infty(x)\{r>t/2\}} \lesssim (\mathcal{E}_2[\phi](T))^{1/2}. \tag{6.38}$$

We can combine these to get our main results:

Lemma 6.4. *For a function ϕ with sufficient decay, we have the estimate*

$$\left\|\tau_+^{s+1}\tau_-^{1/2}\left(\frac{D_{L^*}(r^*\phi)}{r^*}\chi_{\{r^*\ge t/2\}}+L^*(\phi)\chi_{\{r^*\le t/2\}}\right)(w')^{1/2}\right\|_{L^2(t)L^\infty(x)} \lesssim (1+\mathcal{E}_1[F](T)+|q|)(\mathcal{E}_3[\phi](T)). \tag{6.39}$$

Finally, we turn our attention to the pure L^∞ bound on the nice terms. Our analysis is fundamentally based on the estimate (9.19). We first take the reduced light cone energy

$$C_0^*[\phi] = \left\| \left(\tau_+^s \left| \frac{D_{L^*}(r\phi)}{r^*} \right| + \tau_+^s \left| \tau_- \tau_+^{-2} \phi \right| + \tau_-^s \tau_0 |D_{\underline{L}^*} \phi| + \tau_-^s |D_{S^*} \phi| \right) w \right\|_{L^2(C_{u^*})}^2 . \tag{6.40}$$

Recalling the estimate (4.34), we see that the difference between this and $C_0[\phi]$ can be bounded by the integrated second-order energy. In particular, we are integrating a quantity decaying like $M\mathcal{E}_2^{1/2}\tau_+^{-4}\tau_- w$ (for the $\underline{L}^*$ derivative) and like $\tau_+^{2s-6}\tau_-^{3/2-s}\mathcal{E}_2[\phi]w$ (in the undifferentiated case) over the light cone, so this is indeed bounded. We can therefore say:

$$C_l^*[\phi] \lesssim \mathcal{E}_{l+2}[\phi]. \tag{6.41}$$

We can reduce this further by changing the domain of integration and set of null frames in both this and the estimate on F; however, it is not necessary, as the number of derivatives required on the metric for our L^∞ estimates is still the bottleneck.

We start with the conical estimate

$$\left\| \tau_+^{3/2+s} \chi \frac{D_{L^*}(r^*\phi)}{r^*} w^{1/2} \right\|_{L^\infty(C_{u^*})}^2 \lesssim \sum_{\substack{|I|,|J|\leq 1 \\ X \in \{\underline{u}^* L^*, \mathbb{O}, Y \in \mathbb{O}\}}} \left\| \tau_+^s D_X^I D_Y^J \left(\chi \frac{D_{L^*}(r^*\phi)}{r^*} \right) w^{1/2} \right\|_{L^2(C_{u^*})}^2 . \tag{6.42}$$

We can move χ outside the derivatives on the right hand side with no issue, as when derivatives fall on it we get a uniformly bounded quantity. We can rewrite

$$\underline{u}^* L^* = Z^* + \Omega_{0r}^*,$$

and therefore bound by our Lorentz fields.

We again take the commutator estimate (6.34):

$$\sum_{\substack{|I|,|J|\leq 1 \\ X \in \{\underline{u}^* L^*\} \cup \mathbb{O}, Y \in \mathbb{O}}} \left| D_X D_Y \left(\frac{D_{L^*}(r^*\phi)}{r^*} \right) \right| \lesssim \sum_{\substack{|I|\leq 2 \\ X \in \mathbb{L}}} \left| \left(\frac{D_{L^*}(r^* D_X^I \phi)}{r^*} \right) \right| + \sum_{\substack{|I|\leq 1 \\ X \in \mathbb{L}}} \tau_0 \left| \frac{D_X^I \phi}{r^*} \right| +$$
$$+ \sum_{\substack{|I|+|J|\leq 1 \\ X,Y \in \mathbb{L}}} \tau_+ |\alpha(\mathcal{L}_X^I F)||D_Y^J \phi| + \sum_{\substack{|I|+|J|\leq 1 \\ X,Y \in \mathbb{L}}} \tau_- |\rho(\mathcal{L}_X^I F)||D_Y^J \phi|.$$

We can bound this term-by-term. The first term on the right appears in the energy. For the second term on the right, we take the L^∞ estimate

$$\tau_+^s \tau_0 \chi \left| \frac{D_X^I \phi}{r^*} \right| w^{1/2} \lesssim \tau_+^{s-3} \tau_-^{3/2-s} \mathcal{E}_2[\phi]^{1/2}. \tag{6.43}$$

We can take the L^2 norm of the right hand side by direct integration. In particular we have

$$\sum_{\substack{|I| \lesssim 1 \\ X \in \mathbb{L}}} \left\| \tau_+^{s-3} \tau_-^{3/2-s} \mathcal{E}_2[D_X^I \phi]^{1/2} \right\|_{L^2(C_{u^*})}^2 \lesssim \mathcal{E}_3[\phi]. \tag{6.44}$$

For the third term on the right we use

$$\sum_{\substack{|I|+|J| \leq 1 \\ X,Y \in \mathbb{L}}} \left\| \tau_+^{s+1} |\alpha(\mathcal{L}_X^I F)| |D_Y^J \phi| w^{1/2} \right\|_{L^2(C_{u^*})} \lesssim \mathcal{E}_3^{1/2}[\phi] \left\| \tau_+^s \tau_-^{1/2-s} |\alpha(\mathcal{L}_X^I F)| \right\|_{L^2(C_{u^*})}. \tag{6.45}$$

The latter term is easily bounded by our energy. For our fourth term, we first subtract off the charge part, noting that the term containing the charge part is bounded by the second term times the magnitude of the charge. For the remainder, we once again use the integrated L^∞ terms. We have the estimate

$$\sum_{\substack{|I|+|J| \leq 1 \\ X,Y \in \mathbb{L}}} \left\| \tau_+^s \tau_- |\tilde{\rho}(\mathcal{L}_X^I F)| |D_Y^J \phi| w^{1/2} \right\|_{L^2(C_{u^*})} \lesssim \mathcal{E}_3^{1/2}[\phi] \mathcal{E}_3^{1/2}[\widetilde{F}] \left\| \tau_+^{-2} \tau_-^{1-s} \right\|_{L^2(C_{u^*})}. \tag{6.46}$$

The remaining L^2 norm on the right is directly bounded.

Note that this result is significantly easier to show than the corresponding result in [16] since we have extra room in the number of derivatives. We state our result as follows:

Proposition 6.5. *For any function ϕ with sufficient decay, given the estimates $\mathcal{E}_4[F], |q| \leq 1$, we have the bound*

$$\left| \tau_+^{3/2+s} \left| \chi \frac{D_{L^*}(r^* \phi)}{r^*} \right| w^{1/2} \right| \lesssim \mathcal{E}_4^{1/2}[\phi]. \tag{6.47}$$

This follows from the fact that we only need to use the conical energy in the first term on the right. We can combine these estimates as follows.

Theorem 6.6. *Given a function ϕ, and a two-form F satisfying* (1.3)*, with $|q[F]|, \mathcal{E}_3[\widetilde{F}] \leq 1$, we have the* *estimates*

$$\left(\tau_+\tau_-^{s-1/2}|\phi| + \tau_+\tau_-^{1/2+s}|D_{\underline{L}^*}\phi| + \tau_+^{s+1}\tau_-^{1/2}(|D_{S^*}\phi| + |D_{L^*}\phi|)\right) w^{1/2} \lesssim \mathcal{E}_2^{1/2}(T), \tag{6.48}$$

$$\left\|\tau_+^{s+1}\tau_-^{1/2}\left(\frac{D_{L^*}(r^*\phi)}{r^*}\chi_{\{r^*\geq t/2\}} + L^*(\phi)\chi_{\{r^*\leq t/2\}}\right)(w')^{1/2}\right\|_{L^2(t)L^\infty(x)} \lesssim (1+\mathcal{E}_1[F](T)+|q|)(\mathcal{E}_3[\phi](T)). \tag{6.49}$$

$$\tau_+^{s+3/2}\left|\chi\frac{D_{L^*}(r^*\phi)}{r^*}\right| w^{1/2} \lesssim \mathcal{E}_4^{1/2}(T). \tag{6.50}$$

7 Commutator Estimates: F

We first combine our energy and decay estimates into a form which will show what we need to show for the main theorem. Recalling the energy norms

$$\mathcal{E}_0[F](T) = E_0[F](T) + S_0[F](T) + C_0[F](T),$$

$$\mathcal{E}_0[\phi](T) = E_0[\phi](T) + S_0[\phi](T) + C_0[\phi](T),$$

$$\mathcal{E}_k(T) = \mathcal{E}_k[F,\phi](T) = \sum_{|I|\leq k, X\in\mathbb{L}} \mathcal{E}_0[\mathcal{L}_X^I\widetilde{F}](T) + \mathcal{E}_0[D_X^I\phi] + |q[F]|^2,$$

and the initial estimate $\mathcal{E}_k[F](T) \leq 1$, we have the following combined estimate:

Theorem 7.1. *Given (ϕ, F) solving the MKG system for time $[0,T]$ such that $\mathcal{E}_k(T) \leq 1$, $k \geq 5$, we have the following:*

$$\mathcal{E}_k(T) \lesssim \mathcal{E}_k(0) + \sum_{\substack{|I|\leq k\\ X\in\mathbb{L}}} \left\|J[\mathcal{L}_X^I\widetilde{F}]\right\|_{L^2[w]}^2 + \sum_{\substack{|I|\leq k\\ X\in\mathbb{L}}} \left\|(\Box_g^{\mathbb{C}} D_X^I\phi)\tau_+^s\tau_-^{1/2}w_\iota^{1/2}\right\|_2^2. \tag{7.1}$$

In order to prove global existence, we must take three steps: first, we must show that the two spacetime energy norms on the right are bounded by a small constant times the initial energy, in such a way that they can be subtracted off from the left hand sign while keeping positivity, and second, we must show that the first term on the right is bounded by our initial energy norms.

As a matter of convention, we assume $|I| \leq k$; which will in general mean that the total number of Lorentz fields on the left is bounded by the number of derivatives contained in the energy.

We now look at the value of the commutator terms appearing for F, J. We recall the weighted current norm (3.9). It suffices to bound this by the energy for all Lie derivatives of $\widetilde{F}$:

Theorem 7.2. *Given the current norm*

$$\|J\|_{L^2[w]} = \left\|\tau_+^s\tau_0^{-1/2-\iota}\tau_-^{1/2}|J_{L^*}|w_\iota^{1/2}\right\|_2 + \left\|\tau_0^{s-1/2-\iota}\tau_-^{s+1/2}|J_{\underline{L}^*}|w_\iota^{1/2}|\right\|_2 + \left\|\tau_+^s\tau_-^{1/2}|J_{S^*}|w_\iota^{1/2}\right\|_2, \tag{7.2}$$

where (F,ϕ) are solutions to the MKG system in the metric, then we have the estimate

$$\sum_{\substack{|I|\le k\\ X\in\mathbb{L}}} \left\|J[\mathcal{L}_X^I\widetilde{F}]\right\|_{L^2[w]} \lesssim \mathcal{E}_k(T)+\epsilon^2. \tag{7.3}$$

The full power of $\mathcal{E}_k$ (and corresponding ϵ^2) are necessary, as we will use them to bound the right hand side of our energy estimate for F by a small constant times the energy.

Remark. *Here we are again using the modified vector fields $\mathbb{L}$ instead of their Killing and Conformal Killing analogues in Minkowski space, and we can illuminate the reason for their use. In general, we note that for example $\alpha[F]$ has L^∞ bounds which like $\tau_0^{1/2+s}$ times the bounds for $\underline{\alpha}[F]$. The difference is more drastic when we look at our $L^(t)L^\infty(x)$ bounds, as if we try to establish the $L^2(L^\infty)$ bounds with our replaced weights for $\underline{\alpha}$ we have an additional logarithmic growth. However, if we commute $Z=t\partial_t+r\partial_r$ with L^*, and take the corresponding term like $\alpha[\mathcal{L}_ZF]$, we get a term behaving roughly like $\frac{M}{t}\underline{\alpha}[F]$ along the light cone. This in general is not compatible with the decay estimates we require, so the curved fields $\mathbb{L}$ seem to be necessary here, in addition to the the nicer decay estimates they provide for Lie derivatives of g.*

Proof. We recall the identity (2.34), which we can apply the identity (2.23) and iterate over multiple Lie derivatives to get

$$J[\mathcal{L}_X^I\widetilde{F}]^\alpha = (\mathcal{L}_X^I\tilde{J})^\alpha + \sum_{\{X^{I_1},X_1,X^{I_2}\}=X^I} -\frac{1}{2}\mathcal{L}_X^{I_1}\left(\nabla_\beta(g^{\gamma\delta}(\mathcal{L}_Xg)_{\gamma\delta})(\mathcal{L}_X^{I_2\cdot}(F^\dagger))^{\alpha\beta}\right), \tag{7.4}$$

where

$$F^{\dagger\alpha\beta} = \widetilde{F}^{\alpha\beta}.$$

We use this notation in order to reduce ambiguity in raising and lowering indices. In general, we define

$$(\mathcal{L}_X^I\tilde{J})_\alpha = g_{\alpha\beta}(\mathcal{L}_X^I\tilde{J})^\beta.$$

Note that on the right hand side of equation (7.4) we can replace $(\mathcal{L}_Xg)_{\gamma\delta}$ with ${}^{(X)}\tilde{\pi}_{\gamma\delta}$, as the trace of their difference is constant and therefore becomes 0 when differentiated.

We can bound the corresponding current norms in magnitude. There are three parts to this. First, in the first term, $(\mathcal{L}_X^I \tilde{J})^\alpha$, we consider the terms where we take L^2 norms on F or ϕ. These are conceptually very similar to the estimates used in [16] in the following sense: We will often require bilinear estimates on quantities behaving roughly like, e.g.,

$$g^{XY} \frac{D_X(r^*\phi)}{r^*} F_{S_i^* Y}.$$

In the Minkowski case, every time a bad derivative appears on ϕ, it's paired with a nice component of F, and vice versa. In our case, this is not necessarily true, but when bad terms are paired with each other, they are combined with a metric term with decay like $\tau_-^{\gamma'} \tau_+^{-1-\gamma'+\iota}$. This decay rate suffices to bound the corresponding bad terms by the energy, and comes from the improved decay estimates found in [13].

The second term comes from taking L^2 norms on the metric g when taking repeated Lie derivatives of $\tilde{J}^\alpha$. We can first of all subtract off terms like $c_X g$, and replace this with energy norms on our error tensor ${}^{(X)}\tilde{\pi}^\dagger$ combined with L^∞ norms on g. This is of course is not an issue in the Minkowski metric, as our corresponding ${}^{(X)}\tilde{\pi}^\dagger$ is identically 0. Here, we generally have more room since we are dealing with error terms.

Finally, we look at the second term on the right hand side of (7.4). This is again 0 in the Minkowski case, as the divergence of Lorentz vector fields is constant. We use combined estimates; in general, we have a good amount of room here.

We first look at $(\mathcal{L}_X^I \tilde{J})^\alpha$. Decomposing

$$(\mathcal{L}_X^I \tilde{J})^\alpha = (\mathcal{L}_X^I J)^\alpha - (\mathcal{L}_X^I \overline{J})^\alpha, \tag{7.5}$$

it again suffices to bound the current norms for both quantities on the right. We can expand the first term out as

$$(\mathcal{L}_X^I J)^\alpha = \mathcal{L}_X^I \left(g^{\alpha\beta} \Im \left(\phi \overline{D_\beta \phi} \right) \right). \tag{7.6}$$

In each term where the Lie derivative falls on g, or a modified Lie derivative, we take the decomposition into $\widetilde{\mathcal{L}}_X g$ and $c_X g$.

In the case where most derivatives fall on the ϕ terms, our analysis closely matches that in [16]. In the cases where most derivatives fall on the metric, we have an estimate with more room, albeit a more tedious one.

When derivatives fall on g, we want to decompose and iterate using $(\mathcal{L}_X g)^{\alpha\beta} = (\widetilde{\mathcal{L}}_X g)^{\alpha\beta} - c_X g^{\alpha\beta}$, so that the terms where most Lie derivatives fall on the metric can be bounded using our energy estimates.

Using the identity

$$X\left(\phi\overline{\psi}\right) = D_X\phi\overline{\psi} + \phi\overline{D_X\psi}, \tag{7.7}$$

along with the commutator identity

$$D_X D_\alpha\phi = D_\alpha D_X\phi - (\partial_\alpha X^\beta)D_\beta\phi + iF_{X\alpha}\phi \tag{7.8}$$

we can expand

$$\mathcal{L}_X(\phi\overline{D_\alpha\psi}) = D_X\phi\overline{D_\alpha\psi} + \phi\overline{D_\alpha D_X\psi} + iF_{X\alpha}(\phi\overline{\psi}). \tag{7.9}$$

Iterating and symmetrizing this gives us, for any vector Z,

$$\begin{aligned} \left|(\mathcal{L}_X^I(\phi\overline{D\phi}))_Z\right| \lesssim & \sum_{\substack{|I_1|+|I_2|\leq|I| \\ X_i\in\mathbb{L}}} \left|\Im\left(D_{X_1}^{I_1}\phi\overline{D_Z D_{X_2}^{I_2}\phi} + D_{X_2}^{I_2}\phi\overline{D_Z D_{X_1}^{I_1}\phi}\right)\right| + \\ & + \sum_{\substack{|I_1|+|I_2|+|I_3|\leq|I|-1 \\ X_i,Y\in\mathbb{L}}} \left|D_{X_1}^{I_1}\phi\overline{D_{X_2}^{I_2}\phi}|(\mathcal{L}_{X_3}^{I_3}F)_{YZ}\right|. \end{aligned} \tag{7.10}$$

The first term on the right can be fortunately improved in the nice component due to our symmetrization, as we can replace $D_{L^*}\cdot$ with $\frac{D_{L^*}(r^*\cdot)}{r^*}$ The difference is the imaginary part of a real quantity, thus equal to 0.

We now take L^2 and L^∞ quantities on this. The L^∞ estimates are necessary in order to control the metric energy, and are not necessary in the Minkowski case.

First, if $|I|\leq k-6$, we have

$$|\mathcal{L}_X^I J_{L^*}| \lesssim \mathcal{E}_k(T)\tau_+^{-5/2-s}\tau_-^{1/2-s}w^{-1}, \tag{7.11a}$$

$$|\mathcal{L}_X^I J_{S^*}| \lesssim \mathcal{E}_k(T)\tau_+^{-2-s}\tau_-^{-s}w^{-1}, \tag{7.11b}$$

$$|\mathcal{L}_X^I J_{\underline{L}^*}| \lesssim \mathcal{E}_k(T)\tau_+^{-2}\tau_-^{-2s}w^{-1}. \tag{7.11c}$$

We can combine these with energy estimates on the metric. Decomposition is fortunately not necessary here:

$$\sum_{\substack{|I_1|+|I_2|\leq|I| \\ X_i\in\mathbb{L}}} \left\||\mathcal{L}_{X_1}^{I_1}J||^{(X_2^{I_2})}\tilde{\pi}^\dagger|\tau_+^{s+1/2+\iota}\tau_-^{-\iota}w_i^{1/2}\right\|_2 \lesssim \mathcal{E}_k(T)^{1/2}\left\|\tau_+^{s+1/2-\iota-2}\tau_-^{\iota-2s}{}^{(X^I)}\tilde{\pi}^\dagger\right\|_2. \tag{7.12}$$

Using $s+1/2-\iota-2 < -1/2-2\iota$, $\iota-2s<-1$, we can easily bound this by $\mathcal{E}_k(T)^{1/2}\epsilon$.

In the case where we have L^∞ estimates on the metric, we note that we can lower indices easily to require bilinear estimates analgous to those proven in [16], as all error terms correspond with metric terms with decay faster than τ_0^{2s}, the difference in weights between the highest and lowest weights in the norm (7.2).

We first look at all terms in (7.10) where F does not appear. First, in the J_{L^*} term of the norm we need to show

$$\left\|\tau_+^s\tau_0^{-1/2-\iota}\tau_-^{1/2}|(\mathcal{L}_X^I J)_{L^*}|w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k. \tag{7.13}$$

We have the bilinear estimate

$$\sum_{\substack{|I_1|+|I_2|\le|I| \\ X_i\in\mathbb{L}}}\left\|\tau_+^{s+1/2+\iota}\tau_-^{-\iota}D_{X_1}^{I_1}\phi\frac{\overline{D_{L^*}D_{X_2}^{I_2}\phi}}{r^*}w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k(T). \tag{7.14}$$

This follows from taking the L^∞ estimate on whichever term has fewer derivatives applied. We will go through this estimate in detail: first, if $|I_1|\le k-6$, we have

$$\left\|\tau_+^{s+1/2+\iota}\tau_-^{-\iota}D_{X_1}^{I_1}\phi\frac{\overline{D_{L^*}D_{X_2}^{I_2}\phi}}{r^*}w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k^{1/2}\left\|\tau_+^{s-1/2+\iota}\tau_-^{1/2-s-\iota}\frac{D_{L^*}D_{X_2}^{I_2}\phi}{r^*}w_\iota^{1/2}\right\|_2. \tag{7.15}$$

This can be easily bounded by $\mathcal{E}_k$, using the identity $\tau_+^{s-1/2+\iota}\tau_-^{1/2-s-\iota}\le\tau_+^s\tau_-^{-s}$, then using $\tau_-^{-s}w_\iota^{1/2}\lesssim w'^{1/2}$.

If $|I_2|\le k-6$, we can take

$$\left\|\tau_+^{s+1/2+\iota}\tau_-^{-\iota}D_{X_1}^{I_1}\phi\frac{\overline{D_{L^*}D_{X_2}^{I_2}\phi}}{r^*}w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k^{1/2}\left\|\tau_+^{\iota}\tau_-^{-\iota}\frac{D_{X_1}^{I_1}\phi}{\tau_+}w_\iota^{1/2}\right\|_2. \tag{7.16}$$

Using the identity $s-2\iota>1/2$, we can use the inequality $\tau_+^\iota\tau_-^{-\iota}<\tau_0^{1/2+\iota}\tau_+^s\tau_-^{-s}$, which gives us our bounds. In general, we have the estimate

$$\left\|\tau_0^{-\iota}\frac{D_{X_1}^{I_1}\phi}{\tau_+}w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k^{1/2}. \tag{7.17}$$

For the angular derivative term, we need to show

$$\left\|\tau_+^s\tau_-^{1/2}|(\mathcal{L}_X^I J)_{S_i^*}|w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k^{1/2} \tag{7.18}$$

We again look at the terms without F first; here, we need to bound

$$\sum_{\substack{|I_1|+|I_2|\leq|I| \\ X_i\in\mathbb{L}}} \left\|\tau_+^s\tau_-^{1/2}\left|D_{X_1}^{I_1}\phi\right|\left|D_{S_j^*}D_{X_2}^{I_2}\phi\right|w_\iota^{1/2}\right\|_2. \tag{7.19}$$

As usual, if most derivatives appear in I_1, we take the L^∞ bound on the angular derivative which gives us the estimate

$$\mathcal{E}_k^{1/2}\left\|\left|\frac{D_{X_1}^{I_1}\phi}{\tau_+}\right|w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k,$$

which follows from expanding and taking the inequality (7.17). If most derivatives appear in I_2, we need to bound

$$\mathcal{E}_k^{1/2}\left\|\tau_+^s\tau_0\tau_-^{-s}\left|D_{S_j^*}D_{X_1}^{I_2}\phi\right|w_\iota^{1/2}\right\|_2,$$

which is likewise easily bounded by our energy.

Finally, we need to show

$$\left\|\tau_0^{s-1/2-\iota}\tau_-^{s+1/2}|J_{\underline{L}^*}|w_\iota^{1/2}\right\|_2 \lesssim \mathcal{E}_k^2 \tag{7.20}$$

We again consider the noncommutator terms. If most derivatives fall on I_1, we have that this can be bounded by

$$\mathcal{E}_k^{1/2}\left\|\tau_0^{s-1/2-\iota}\left|\frac{D_{X_1}^{I_1}\phi}{\tau_+}\right|w_\iota^{1/2}\right\|_2,$$

which is an almost immediate consequence of (7.17). If most derivatives fall on I_2, we have to bound

$$\mathcal{E}_k^{1/2}\left\|\tau_0^{s+1/2-\iota}|D_{\underline{L}^*}D_{X_2}^{I_2}\phi|w_\iota^{1/2}\right\|_2.$$

The inequalities $\tau_0^{s+1/2-\iota}\leq\tau_0^{1/2+\iota}$ and $1\lesssim\tau_-^s\tau_-^{-1/2-\iota}$ give us the desired energy bound.

Now we look at the terms containing F. We first look at the charged portion of F, which we can treat as roughly the same in all components. For Lorentz fields Y we recall the estimates

$$\left|(\mathcal{L}_X^I\overline{F})_{YL^*}\right| \lesssim \mathcal{E}_k^{1/2}(T)\tau_+^{-2}\tau_-, \tag{7.21a}$$

$$\left|(\mathcal{L}_X^I\overline{F})_{YS_j^*}\right| \lesssim \mathcal{E}_k^{1/2}(T)\tau_+^{-1}, \tag{7.21b}$$

$$\left|(\mathcal{L}_X^I\overline{F})_{Y\underline{L}^*}\right| \lesssim \mathcal{E}_k^{1/2}(T)\tau_+^{-1}. \tag{7.21c}$$

Using our usual L^∞ estimate on terms where fewer derivatives fall on ϕ (without loss of generality we assume this consists of the I_1 derivatives), and noting the identity $w_\iota \approx \tau_- w'$ in the region where $\overline{F}$ is supported we have that the part of the current norm coming from these terms can be bounded by

$$\mathcal{E}_k^{1/2} \sum_{\substack{|I_1|+|I_2|\leq|I| \\ X_i\in\mathbb{L}}} \left\| \tau_+^{s-1}\tau_- |D_{X_1}^{I_1}\phi||D_{X_2}^{I_2}\phi|(w')^{1/2}w^{1/2} \right\|_2, \tag{7.22}$$

thus, without loss of generality, by

$$\mathcal{E}_k(T) \left\| \tau_+^{s-2}\tau_-^{3/2-s} |D_{X_2}^{I_2}\phi|(w')^{1/2} \right\|_2. \tag{7.23}$$

Taking the inequality $\tau_+^{-1}\tau_-^{3/2-s} \leq \tau_0^{1/2+\iota}$ and writing this in the form of our energy norm gives us our bound.

For the uncharged portion of F, we can again handle all terms similarly in the case when $k-6$ or fewer derivatives fall on $\widetilde{F}$. We take the estimates, for $|I| \leq k-6$,

$$(\mathcal{L}_X^I \widetilde{F})_{YL^*} \lesssim \mathcal{E}_k^{1/2}(T)\tau_+^{-1/2-s}, \tag{7.24a}$$

$$(\mathcal{L}_X^I \widetilde{F})_{YS_j^*} \lesssim \mathcal{E}_k^{1/2}(T)\tau_+^{-s}\tau_-^{-1/2}, \tag{7.24b}$$

$$(\mathcal{L}_X^I \widetilde{F})_{Y\underline{L}^*} \lesssim \mathcal{E}_k^{1/2}(T)\tau_-^{-1/2-s}. \tag{7.24c}$$

Plugging each of these into the current norm, we see that it suffices to show the uniform estimate

$$\sum_{\substack{|I_1|+|I_2|\leq|I| \\ X_i\in\mathbb{L}}} \left\| \tau_0^{-\iota} |D_{X_1}^{I_1}\phi||D_{X_2}^{I_2}\phi| w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k(T). \tag{7.25}$$

This follows from taking our L^∞ norm on whichever term fewer derivatives fall on, and subsequently using the estimate (7.17), with $\tau_-^{1/2-s} w_\iota^{1/2} \lesssim s^{1/2}$.

Finally, we look at when the most derivatives fall on $\widetilde{F}$. We have here the uniform L^∞ estimates on ϕ, so we need to show the bounds

$$\left\| \tau_+^{s-2}\tau_0^{-1/2-\iota}\tau_-^{3/2-2s} (\mathcal{L}_X^I \widetilde{F})_{YL^*} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2}, \tag{7.26a}$$

$$\left\| \tau_+^{s-2}\tau_-^{3/2-2s} (\mathcal{L}_X^I \widetilde{F})_{YS_j^*} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2}, \tag{7.26b}$$

$$\left\| \tau_+^{-2}\tau_0^{s-1/2-\iota}\tau_-^{3/2-s} (\mathcal{L}_X^I \widetilde{F})_{Y\underline{L}^*} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2}. \tag{7.26c}$$

In each case, we expand Y in the null frame.

We can bound the first term by

$$\left\| \tau_+^s \tau_0^{1/2-\iota} \tau_-^{1/2-2s} |\alpha[\mathcal{L}_X^I \widetilde{F}]| w_\iota^{1/2} \right\|_2 + \left\| \tau_+^s \tau_0^{3/2-\iota} \tau_-^{1/2-2s} |\rho[\mathcal{L}_X^I \widetilde{F}]| w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2}.$$

Our energy norm follows from taking the inequalities $\tau_0^{1/2-\iota} \leq 1$, $\tau_0^{3/2-\iota} \leq \tau_0^{1/2+\iota}$, $\tau_-^{1/2-2s} \leq \tau_-^{-1/2-\iota}$.

We can expand the second term by

$$\left\| \tau_+^s \tau_0^1 \tau_-^{1/2-2s} (|\alpha[\mathcal{L}_X^I \widetilde{F}]| + |\sigma[\mathcal{L}_X^I \widetilde{F}]|) w_\iota^{1/2} \right\|_2 + \left\| \tau_0^{2-s} \tau_-^{1/2-s} |\underline{\alpha}[\mathcal{L}_X^I \widetilde{F}]| w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2}.$$

Again, we have our energy norm, which follows from $\tau_0 \leq \tau_0^{1/2+\iota}$, $\tau_0^{2-s} \leq \tau_0^{-1/2-\iota}$, $\tau_-^{1/2-s} \leq \tau_-^s \tau_-^{-1/2-\iota}$.

Finally, the third term can be bounded by

$$\left\| \tau_0^{s+1/2-\iota} \tau_-^{1/2-s} (|\underline{\alpha}[\mathcal{L}_X^I \widetilde{F}]| + |\rho[\mathcal{L}_X^I \widetilde{F}]|) w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2}$$

We have $\tau_0^{s+1/2-\iota} \leq \tau_0^{1/2+\iota}$, $\tau_-^{1/2-s} \leq \tau_-^s \tau_-^{-1/2-\iota}$, which lead to our energy bounds.

Therefore, we have the estimate

$$\left\| (\mathcal{L}_X^I J) \right\|_{L^2[w]} \lesssim \mathcal{E}_k(T) + \epsilon^2. \tag{7.27}$$

We consider the $\overline{J}^\alpha$ term. Since F is antisymmetric, we have the identity

$$\nabla_\beta \overline{F}^{\alpha\beta} = \frac{1}{\sqrt{|g|}} \partial_\beta (\sqrt{|g|} \cdot \overline{F}^{\alpha\beta}). \tag{7.28}$$

In general it is easier to expand everything out in this form.

When the derivative falls on $\sqrt{|g|}$, we have the equation

$$\sum_{\substack{|I| \leq k \\ X \in \mathbb{L}}} \mathcal{L}_X^I \left(\frac{\partial_\beta \sqrt{|g|}}{\sqrt{|g|}} \overline{F}^{\alpha\beta} \right) \lesssim \sum_{\substack{|I_1|+|I_2| \leq k \\ X_i \in \mathbb{L}}} \left| \partial_\beta \left(X_1^{I_1} \ln(|g|) \right) \mathcal{L}_{X_2}^{I_2} \overline{F}^{\alpha\beta} \right|. \tag{7.29}$$

We expand $\partial_\beta \left(X_1^{I_1} \ln(|g|) \right)$, as usual first when up to $k-6$ derivatives fall on $|g|$, then when up to k

derivatives fall in it. When $|I_1| \leq k-6$,, we have the straightforward estimates

$$L^*\left(X_1^{I_1}\ln(|g|)\right) \lesssim \epsilon\tau_+^{-2+\iota}, \tag{7.30a}$$

$$S_j^*\left(X_1^{I_1}\ln(|g|)\right) \lesssim \epsilon\tau_+^{-2+\iota}, \tag{7.30b}$$

$$\underline{L}^*\left(X_1^{I_1}\ln(|g|)\right) \lesssim \epsilon\tau_+^{-1+\iota}\tau_-^{-1}. \tag{7.30c}$$

When $|I| \leq k$, we have the energy estimates

$$\left\|\tau_-^{-1/2}\tau_+^{-\iota}L^*\left(X_1^{I_1}\ln(|g|)\right)(1+\tau_-^{1/2}\overline{\chi})\right\|_2 \lesssim \epsilon \tag{7.31a}$$

$$\left\|\tau_-^{-1/2}\tau_+^{-\iota}S_i^*\left(X_1^{I_1}\ln(|g|)\right)(1+\tau_-^{1/2}\overline{\chi})\right\|_2 \lesssim \epsilon \tag{7.31b}$$

$$\left\|\tau_+^{-1/2-\iota}\underline{L}^*\left(X_1^{I_1}\ln(|g|)\right)(1+\tau_-^{1/2}\overline{\chi})\right\|_2 \lesssim \epsilon \tag{7.31c}$$

These come from the fact that the worst decay happens when all derivatives fall on the same quantity. Now we look at Lie derivatives on $\overline{F}^{\alpha\beta}$. Again, when fewer than $k-6$ derivatives fall on $\overline{F}$ we have

$$|(\mathcal{L}_{X_2}^{I_2}\overline{F})^{L^*S_j^*}| \lesssim q\tau_+^{-2}\overline{\chi}, \tag{7.32a}$$

$$|(\mathcal{L}_{X_2}^{I_2}\overline{F})^{S_i^*S_j^*}| \lesssim q\tau_+^{-2}\overline{\chi}, \tag{7.32b}$$

$$|(\mathcal{L}_{X_2}^{I_2}\overline{F})^{L^*\underline{L}^*}| \lesssim q\tau_+^{-2}\overline{\chi}, \tag{7.32c}$$

$$|(\mathcal{L}_{X_2}^{I_2}\overline{F})^{\underline{L}^*S_j^*}| \lesssim q\tau_+^{-3}\tau_-\overline{\chi}. \tag{7.32d}$$

When up to k derivatives fall on $\overline{F}$, we break up the Lie derivatives falling on g using the modified Lie derivative $\overline{\mathcal{L}}$. In the case where fewer than $k-6$ modified Lie derivatives fall on both metric terms, we can use our L^∞ estimates on $\overline{F}^{\alpha\beta}$ as well as our energy estimates on other metric terms.

We can establish the necessary estimates. First, we look at the $J^{\underline{L}^*}$ terms in the current norm, which intuitively requires the most decay. We have the estimate for $|I_2| \leq k-6$:

$$\sum_{\substack{|I_1|+|I_2|\leq k\\ X_i\in\mathbb{L}}}\left\|\tau_+^s\tau_0^{-1/2-\iota}\tau_-^{1/2}\left|\partial_\beta\left(X_1^{I_1}\ln(|g|)\right)\mathcal{L}_{X_2}^{I_2}\overline{F}^{\alpha\beta}\right|w_\iota^{1/2}\right\|_2 \lesssim |q|\left\|\tau_+^{s+1/2+\iota-2}\tau_-^{-\iota+\delta}\partial\left(X_1^{I_1}\ln(|g|)\right)\right\|_2. \tag{7.33}$$

The τ_- term must be absorbed in our metric norm, using the estimate $\delta \leq 1/2$. Therefore, we can use the estimate $\tau_+^{s+1/2+\iota-2} \leq \tau_+^{-1/2-\iota}$ to bound this by our energy. Note that we do not need to worry about the spherically symmetric perturbation of g, as its derivative uniformly decays like τ_+^{-2} and therefore we have the spacetime norm of something decaying faster than $\epsilon\tau_+^{-2-\iota}$.

When most Lie derivatives falls on a g term in F, we can take similar uniform estimates. This is easier than the previous result, due to the addition L^∞ norm of g, which gives us an extra power of $\tau_+^{-1+\iota}\tau_-^{-1}$ at the cost of having to use the energy coming from our Hardy estimate.

When the derivative passes through $\sqrt{|g|}$, we decompose

$$\overline{F}^{\alpha\beta} = (\tilde{m}^{\alpha\gamma} + H^{\alpha\gamma})(\tilde{m}^{\beta\delta} + H^{\beta\delta})\overline{F}_{\gamma\delta}.$$

We expand and consider two cases: first, when a factor of H appears, and second, when H does not appear. We deal with the latter first, as it is well-defined and is analogous to a term appearing in the Minkowski case [16], and which we will call $\overline{J}_A$:

$$\begin{aligned}
\partial_\beta\left(\tilde{m}^{0\gamma}\tilde{m}^{\beta\delta}\overline{F}_{\gamma\delta}\right) &= \sum_k \partial_k\left(\left(1-\left(\frac{M\chi}{1+r}\right)^2\right)\overline{F}_{0k}\right) &\text{(7.34a)}\\
&= \left(1-\left(\frac{M\chi}{1+r}\right)^2\right)\left[\left(\frac{r^*}{r}-\partial_r(r^*)\right)\left(\frac{q}{2\pi}\frac{\overline{\chi}(r^*-t-2)}{r^{*3}}\right)+\frac{q}{4\pi}\frac{\overline{\chi}'(r^*-t-2)\partial_r(r^*)}{r^{*2}}\right]-\\
&\quad -\partial_r\left(\frac{M\chi}{1+r}\right)^2\frac{q}{4\pi}\frac{\overline{\chi}(r^*-t-2)}{r^{*2}},\\
\partial_\beta\left(\tilde{m}^{j\gamma}\tilde{m}^{\beta\delta}\overline{F}_{\gamma\delta}\right) &= \partial_0\left(\left(1-\left(\frac{M\chi}{1+r}\right)^2\right)\overline{F}_{j0}\right) &\text{(7.34b)}\\
&= \left(-1+\left(\frac{M\chi}{1+r}\right)^2\right)(\partial_t(r^*))\left[\frac{-\omega_j q}{2\pi}\frac{\overline{\chi}(r^*-t-2)}{r^{*3}}+\frac{q\omega_j}{4\pi}\frac{\overline{\chi}'(r^*-t-2)}{r^{*2}}\right]-\\
&\quad -\left(-1+\left(\frac{M\chi}{1+r}\right)^2\right)\frac{q\omega_j}{4\pi}\frac{\overline{\chi}'(r^*-t-2)}{r^{*2}}+\partial_t\left(\left(\frac{M\chi}{1+r}\right)^2\right)\frac{q\omega_i}{4\pi}\frac{\overline{\chi}(r^*-t-2)}{r^{*2}}.
\end{aligned}$$

We can simplify the ∂_t terms greatly, as the support of $\partial_t\chi, \partial_t(r^*)$ are both disjoint to the support of $\overline{\chi}$. In general, we can replace

$$\begin{aligned}
\partial_\beta\left(\tilde{m}^{\alpha\gamma}\tilde{m}^{\beta\delta}\overline{F}_{\gamma\delta}\right) = &\left(1-\left(\frac{M}{1+r}\right)^2\right)\left(1+\frac{M}{1+r}\right)\frac{q}{4\pi}\frac{\overline{\chi}'(r^*-t-2)}{r^{*2}}L^{*\alpha}+ \qquad\text{(7.35)}\\
&+\frac{q}{4\pi}\left[\left(1-\left(\frac{M\chi}{1+r}\right)^2\right)\left(\frac{r^*}{r}-\partial_r(r^*)\right)\left(\frac{2}{r^{*3}}\right)-\partial_r\left(\frac{M\chi}{1+r}\right)^2\frac{1}{r^{*2}}\right]\overline{\chi}(r^*-t-2)\partial^\alpha_{t^*}.,
\end{aligned}$$

Note that the first term, containing L^α, has support in $\tau_-\approx 1$, and the second term, which does not appear in the Minkowski case, decays like $\tau_+^{-4}\ln(\tau_+)$. We now establish estimates on Lie derivatives: In this region, we have that

$$\partial_{r^*}^k(r) = \begin{cases} O(1) & k = 1, \\ O(r^{-k}) & k > 1. \end{cases} \tag{7.36}$$

Therefore, for any number of Lie derivatives of J_A, we have the null decomposition

$$\overline{J}_A^{L^*} \lesssim q\tau_+^{-2}\chi_{\{2\leq r^*-t\leq 3\}} + qM\tau_+^{-4+\iota}\overline{\chi}, \tag{7.37a}$$

$$\overline{J}_A^{S_i^*} \lesssim q\tau_+^{-3}\chi_{\{2\leq r^*-t\leq 3\}} + qM\tau_+^{-4+\iota}\overline{\chi}, \tag{7.37b}$$

$$\overline{J}_A^{\underline{L}^*} \lesssim q\tau_+^{-4}\chi_{\{2\leq r^*-t\leq 3\}} + qM\tau_+^{-4+\iota}\overline{\chi}, \tag{7.37c}$$

where $\chi_{\{2\leq r^*-t\leq 3\}}$ is the characteristic function of the support of $\overline{\chi}'$. Note that in this region $\tau_- \lesssim 1$. It is first easy to see that the current norm of the error terms $qM\tau_+^{-4+\iota}\overline{\chi}$ is bounded above by a constant (determined by s, δ, ι) times qM, which follows from expanding and taking the weight consideration $s+\delta+2\iota < 3/2$. For all other terms, we likewise have the nice bounds

$$\left\|\overline{J}_A\right\|_{L^2[w]}^2 \lesssim |q|^2. \tag{7.38}$$

Now we look at the terms in our decomposition of $\overline{F}^{\alpha\beta}$ containing H. We fortunately can take L^∞ estimates on $\tilde{m}$ and $\overline{F}$, leaving us with energy estimates on H. We can commute everything back through again using (2.34), with ∂_β substituted for ∇_β, as the remainder tensor

$$F_R^{\alpha\beta} = \overline{F}^{\alpha\beta} - \tilde{m}^{\alpha\gamma}\tilde{m}^{\beta\delta}\overline{F}_{\gamma\delta}$$

is also antisymmetric. Therefore, we can say

$$|\left(\mathcal{L}_X^I(\partial_\beta R^{\alpha\beta})\right)| \lesssim |\partial_\beta(\mathcal{L}_Z R)^{\alpha\beta}| + |\partial_\beta(Z^{I_1}(\partial_\gamma Z^\gamma))(\mathcal{L}_{Z_2}^{I_2} R)^{\alpha\beta}|. \tag{7.39}$$

We take the uniform estimate

$$|\left(\mathcal{L}_X^I(\partial_\beta R^{\alpha\beta})\right)| \lesssim |q|\left(|\partial H|\tau_+^{-2} + |H|(\tau_+^{-2}\tau_-^{-1} + \tau_+^{-3+\iota})\right) \tag{7.40}$$

This follows from straightforward expansion, examining what the derivative falls on, and noting that

$$|\partial_\beta(Z^{I_1}(\partial_\gamma Z^\gamma))| < \epsilon\tau_+^{-1+\iota}.$$

This concludes the treatment of the $\overline{J}^\alpha$ terms.

For the commutator term on the right of equation (7.4), we first recall the identity

$$\nabla_\beta(g^{\gamma\delta}(\mathcal{L}_X g)_{\gamma\delta}) = \nabla_\beta(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta}).$$

It follows that

$$\mathcal{L}_{X^I}\nabla_\beta(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta}) \lesssim \sum_{\substack{|J|\leq|I|+1\\ X\in\mathbb{L}}} \nabla_\beta(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X^J g)_{\gamma\delta}) + \sum_{\substack{|J|+|K|\leq|I|+1\\ X\in\mathbb{L}}} \nabla_\beta((\widetilde{\mathcal{L}}_X^J g)^{\gamma\delta}(\widetilde{\mathcal{L}}_X^K g)_{\gamma\delta}). \tag{7.41}$$

Therefore, we have the following estimates: first, if $|I|+1 \leq k-6$, and if the energy corresponding to the metric is $< \epsilon$,

$$|L^{*\alpha}\mathcal{L}_X^I\nabla_\alpha(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta})| \lesssim \epsilon\tau_+^{-2+\iota}, \tag{7.42a}$$

$$|S_i^{*\alpha}\mathcal{L}_X^I\nabla_\alpha(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta})| \lesssim \epsilon\tau_+^{-2+\iota}, \tag{7.42b}$$

$$|\underline{L}^{*\alpha}\mathcal{L}_X^I\nabla_\alpha(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta})| \lesssim \epsilon\tau_-^{-1}\tau_+^{-1+\iota}. \tag{7.42c}$$

If $|I|+1 \leq k$, we have the additional energy estimates

$$\left\|\tau_+^{-\iota}\tau_-^{-1/2}L^{*\alpha}\mathcal{L}_X^I\nabla_\alpha(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta})\right\|_2 \lesssim \epsilon, \tag{7.43a}$$

$$\left\|\tau_+^{-\iota}\tau_-^{-1/2}S_i^{*\alpha}\mathcal{L}_X^I\nabla_\alpha(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta})\right\|_2 \lesssim \epsilon, \tag{7.43b}$$

$$\left\|\tau_+^{-1/2-\iota}\underline{L}^{*\alpha}\mathcal{L}_X^I\nabla_\alpha(g^{\gamma\delta}(\widetilde{\mathcal{L}}_X g)_{\gamma\delta})\right\|_2 \lesssim \epsilon. \tag{7.43c}$$

A similar case happens for the terms where Lie derivatives fall on both terms, noting that we must consider the case where the derivative falls on the g term with most modified Lie derivatives or not. Intuitively, by our Hardy estimates (9.30a)-(9.30b), this does not change anything, as we have an extra power of τ_+^{-1} in the L^∞ weight to compensate for the lost derivative.

Now we look at the case when Lie derivatives fall on $F^\dagger$. First, if $k-6$ or fewer derivatives fall on $F^\dagger$, we have the following:

$$\sum_{\substack{|J|\leq k-6\\ X\in\mathbb{L}}} (\mathcal{L}_X^J F^\dagger)^{L^*\underline{L}^*} \lesssim \tau_+^{-s-1}\tau_-^{-1/2+\iota} w_\iota^{-1/2}(\mathcal{E}_k+\epsilon), \tag{7.44a}$$

$$\sum_{\substack{|J|\leq k-6\\ X\in\mathbb{L}}} (\mathcal{L}_X^J F^\dagger)^{L^* S^*} \lesssim \tau_+^{-1}\tau_-^{-1/2-s+\iota} w_\iota^{-1/2}(\mathcal{E}_k+\epsilon), \tag{7.44b}$$

$$\sum_{\substack{|J|\leq k-6\\ X\in\mathbb{L}}} (\mathcal{L}_X^J F^\dagger)^{\underline{L}^* S^*} \lesssim \tau_+^{-3/2-s}\tau_-^{\iota} w_\iota^{-1/2}(\mathcal{E}_k+\epsilon), \tag{7.44c}$$

$$\sum_{\substack{|J|\leq k-6\\ X\in\mathbb{L}}} (\mathcal{L}_X^J F^\dagger)^{S_1^* S_2^*} \lesssim \tau_+^{-s-1}\tau_-^{1/2+\iota} w_\iota^{-1/2}(\mathcal{E}_k+\epsilon), \tag{7.44d}$$

$$\tag{7.44e}$$

that is, the same L^∞ estimates as the original (unraised) remainder field. This follows from the estimate

$$w^{-1/2} \lesssim \tau_-^{\iota} w_\iota^{-1/2}$$

combined with our L^∞ derivatives.

We must be slightly more careful with the energy, as we must deal with the case where most derivatives fall on the metric. We first expand the Lie derivatives in g in the format of our modified Lie derivatives, $\widetilde{\mathcal{L}}_X$. Then we have

$$\sum_{\substack{|J|\leq k\\ X\in\mathbb{L}}} \left\| \tau_+^s \tau_0^{1/2+\iota} \tau_-^{-1/2-2\iota} (\mathcal{L}_X^J F^\dagger)^{L^*\underline{L}^*} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k+\epsilon, \tag{7.45a}$$

$$\sum_{\substack{|J|\leq k\\ X\in\mathbb{L}}} \left\| \tau_-^{s-1/2-2\iota} \tau_0^{1/2+\iota} (\mathcal{L}_X^J F^\dagger)^{L^* S_i^*} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k+\epsilon, \tag{7.45b}$$

$$\sum_{\substack{|J|\leq k\\ X\in\mathbb{L}}} \left\| \tau_+^s \tau_0^{1/2+\iota} \tau_-^{-1/2-2\iota} (\mathcal{L}_X^J F^\dagger)^{S_1^* S_2^*} w_\iota^{1/2} \right\| \lesssim \mathcal{E}_k+\epsilon, \tag{7.45c}$$

$$\sum_{\substack{|J|\leq k\\ X\in\mathbb{L}}} \left\| \tau_+^s \tau_-^{-1/2-2\iota} (\mathcal{L}_X^J F^\dagger)^{\underline{L}^* S_i^*} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k+\epsilon. \tag{7.45d}$$

Intuitively, these match the energy norms for the lower indices, with some extra negative power in τ_- to control the weight. This holds because when we expand $\dagger F^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} \widetilde{F}_{\gamma\delta}$, when we take the error terms coming from the modified Lie derivatives acting on g, we have two small terms (i.e., one term like $\widetilde{F}$ and one error term h in the metric), while when they act on F, we have one small term. We are now ready to bound the current norm. If we expand the commutator terms in our null frame, we have nothing to worry about. Here I bound the terms corresponding to the "bad" component J^{L^*}; the rest follow similarly. First we take the null decomposition of terms appearing in the commutator, which can be bounded by

$$\sum_{\substack{|J|\leq k-6\\ X\in\mathbb{L}}} |(\mathcal{L}_X^J F^\dagger)^{L^*\underline{L}^*}| \sum_{\substack{|J|+|K|\leq k\\ X\in\mathbb{L}}} \underline{L}^*((\widetilde{\mathcal{L}}_X^J g)^{\gamma\delta}(\widetilde{\mathcal{L}}_X^K g)_{\gamma\delta}) + \sum_{\substack{|J|\leq k-6\\ X\in\mathbb{L}}} |(\mathcal{L}_X^J F^\dagger)^{L^*S^*}| \sum_{\substack{|J|+|K|\leq k\\ X\in\mathbb{L}}} S^*((\widetilde{\mathcal{L}}_X^J g)^{\gamma\delta}(\widetilde{\mathcal{L}}_X^K g)_{\gamma\delta}) +$$
$$\sum_{\substack{|J|\leq k\\ X\in\mathbb{L}}} |(\mathcal{L}_X^J F^\dagger)^{L^*\underline{L}^*}| \sum_{\substack{|J|+|K|\leq k-6\\ X\in\mathbb{L}}} \underline{L}^*((\widetilde{\mathcal{L}}_X^J g)^{\gamma\delta}(\widetilde{\mathcal{L}}_X^K g)_{\gamma\delta}) + \sum_{\substack{|J|\leq k\\ X\in\mathbb{L}}} |(\mathcal{L}_X^J F^\dagger)^{L^*S^*}| \sum_{\substack{|J|+|K|\leq k-6\\ X\in\mathbb{L}}} S^*((\widetilde{\mathcal{L}}_X^J g)^{\gamma\delta}(\widetilde{\mathcal{L}}_X^K g)_{\gamma\delta}).$$

The weights appearing in the J^{L^*} current norm can be bounded above by $\tau_-^{2s} w^{1/2}$. We have room to spare, so this simplifies our calculations: in particular, in the first two terms we can bound the F term by the L^∞ norm

$$|\mathcal{L}_X^J F| w^{1/2} \lesssim \mathcal{E}_k^{1/2} \tau_-^{-s-1/2} \tau_+^{-1},$$

which we can substitute to get

$$\left\| \tau_-^{s-1/2} \tau_+^{-1} \partial((\widetilde{\mathcal{L}}_X^J g)^{\gamma\delta}(\widetilde{\mathcal{L}}_X^K g)_{\gamma\delta}) \right\| \lesssim \mathcal{E}_k^{1/2} \epsilon.$$

We can write the first term like $\tau_0^{s-1/2} \tau_+^{-3/2+s}$, after which our energy bound follows. For the terms where most derivatives fall on the $\widetilde{F}$ terms, we have the (non-decomposed) bounds

$$|\partial((\widetilde{\mathcal{L}}_X^J g)^{\gamma\delta}(\widetilde{\mathcal{L}}_X^K g)_{\gamma\delta})| \lesssim \epsilon \tau_+^{-1+\iota} \tau_-^{-1}.$$

We can combine this with the inequality

$$\left\| \tau_-^{2s-1} \tau_+^{-1+\iota} |\mathcal{L}_X^J \widetilde{F}| w^{1/2} \right\|_2 \lesssim \left\| \tau_-^{s} \tau_+^{-1/2-\iota} |\mathcal{L}_X^J \widetilde{F}| w^{1/2} \right\|_2, \tag{7.46}$$

which gives us the desired energy norm.

We now take the $J^{S_j^*}$ norm. We can think of this in our decomposition as $|\alpha||\partial g| + |F||\bar{\partial} g|$. The current norm weights are $\tau_+^s \tau_-^{1/2} s_\iota^{1/2}$. Straightforward application gives us our desired results, again with close to a power of τ_+ to spare.

Finally, we take the $J^{\underline{L}^*}$ terms, which we can decompose as $|\not{F}||\bar{\partial} g|$ (with appropriate modified Lie derivatives taken). We again bound it using the simpler weight $\tau_+^{2s} w_\iota^{1/2}$, from which our results follow. $\square$

7.1 The Initial Conditions

Before we conclude this, we must show that the initial conditions for our energy estimates are compatible with our initial conditions. To be precise, we need to show the estimate

$$\mathcal{E}_k(0) \lesssim \|E_{0df}\|^2_{H^{k,s_0}(\mathbb{R}^3)} + \|B_0\|^2_{H^{k,s_0}(\mathbb{R}^3)} + \|D\phi_0\|^2_{H^{k,s_0}(\mathbb{R}^3)} + \left\|\dot{\phi}_0\right\|^2_{H^{k,s_0}(\mathbb{R}^3)} + \epsilon^2. \tag{7.47}$$

This largely follows the proof in the Minkowski case, with some adaptations made to account for the metric. It is important to note that B_0 is unaffected by the subtraction of the curl-free part of E_0, even when we raise and lower indices. This follows from the fact that the metric is initially split; i.e. $g_{0i} = 0$.

We first have the charge estimate

$$|q| \leq \left\|(1 + r^*)^{s+\delta} J^0\right\|_{L^{6/5}(x)}, \tag{7.48}$$

which follows from the definition of q along with Hölder's inequality, as $(1 + r^*)^{-s-\delta}$ is bounded in $L^6(x)$.

Additionally, we wish to show the estimate

$$\int_{\Sigma_0} (|E[\overline{F}] - E_{cf}|(1 + r^*)^{s+\delta})^2 \lesssim |q|^2. \tag{7.49}$$

This proceeds virtually identically to the case in Minkowski space. We first cite Lemma 10.1 in [16], which gives the inequality

$$\int_{\mathbb{R}^3} r^{2\delta} \left|\nabla\left(\Delta^{-1}(\sqrt{|g|}J^0) - \frac{q}{4\pi r}\right)\right|^2 dx \lesssim \left\|r^\delta(\sqrt{|g|}J^0)\right\|^2_{L^{6/5}(x)}, \tag{7.50}$$

where $1/2 < \delta < 3/2$ and q is the charge associated with ψ. We set $\psi = \sqrt{|g|}J^0$. We have the consequent inequality

$$\sum_{|I|\leq k} \int_{\mathbb{R}^3} r^{2\delta+2|I|} \left|\nabla\nabla_x^I\left(\Delta^{-1}(\sqrt{|g|}J^0) - \frac{q}{4\pi r}\right)\right|^2 dx \lesssim \sum_{|I|\leq k} \left\|r^{\delta+|I|}\nabla_x^I(\sqrt{|g|}J^0)\right\|^2_{L^{6/5}(x)}. \tag{7.51}$$

These estimates are direct consequences of elliptic estimates which can be found in for instance [16] and we do not include them here. We note that the charge q is associated to the quantity $-(\sqrt{|g|}J^0)$, which accounts for the sign change from the previous paper.

In brief, the first estimate is a modified Sobolev embedding where the term with the worst decay is subtracted off, and the second is a consequent inequality.

Note that we can replace r with r^* using the inequality

$$\left|\nabla\left(\frac{q}{4\pi r}-\frac{q}{4\pi r^*}\right)\right| \lesssim |q|(1+r^*)^{-3+\iota}, \tag{7.52}$$

for any $\iota > 0$, and higher derivatives follow similarly. We can similarly add the $\partial_r(r^*)$ factor. and the $\overline{\chi}$ terms.

Here we use the estimate $-3+\iota+s+\delta < -3/2$, and in all cases we can consequently directly integrate in space. We can rewrite this as

$$\left\|r^\delta\left(E^i_{cf}[F]-\overline{F}_{0i}\right)\right\|_{L^2(x)} \lesssim \left\|r^\delta(\sqrt{|g|}J^0)\right\|_{L^{6/5}(x)}, \tag{7.53a}$$

$$\left\|r^\delta\left(E^i_{cf}[\nabla^I_x F]-\nabla^I_x\overline{F}_{0i}\right)\right\|_{L^2(x)} \lesssim \left\|r^\delta\nabla^I_x(\sqrt{|g|}J^0)\right\|_{L^{6/5}(x)}. \tag{7.53b}$$

Therefore, these quantities satisfy the same charge bounds, so we can for the most part replace them in each case when they occur.

We can use fixed-time estimates in order to recover equation (7.49). It follows that

$$\int_{\Sigma_0}|E[\widetilde{F}]|^2(1+r^*)^{2(s+\delta)}\,dx \lesssim \int_{\Sigma_0}|E_{df}|^2(1+r^*)^{2(s+\delta)}\,dx + \left\|(1+r^*)^{s+\delta}J^0\right\|^2_{L^{6/5}(x)}. \tag{7.54}$$

We now show a similar result for electric and magnetic components of $\mathcal{L}^I_X F$. First, at time 0, we can replace ∂_α with $\partial_{x^{*\alpha}}$, with equivalent norms (note that this is because $\partial_t = \partial_{t^*}$ at time 0). It follows that we can use the nicer commutation relations between our Lorentz fields and the $\partial_{x^{*\alpha}}$. We can write in particular

$$\sum_{|I|\le k, X\in\mathbb{L}}|(\mathcal{L}^I_X\widetilde{F})(\partial_{x^{*\alpha}},\partial_{x^{*\beta}})| \lesssim \sum_{|I|\le k}(1+r^*)^{|I|}\nabla^I_{x,t}(\widetilde{F}). \tag{7.55}$$

Furthermore, this is equivalent to the same quantity with all $\partial_{x^{*\alpha}}$ replaced with ∂_α.

Now we must get rid of time derivatives in F. Our two main tools for this are the Bianchi identity

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0$$

when the derivative falls on the magnetic field, as well as

$$g^{\alpha\gamma}\left(\partial_\alpha F_{\beta\gamma} + \Gamma^\delta_{\alpha\beta} F_{\delta\gamma}\right) = J_\beta$$

when the derivative falls on the electric field. The latter equation can be simplified as follows. We are only considering this when we have the term

$$g^{00}\partial_0 F_{i0} + \Gamma^\delta_{0i} F_{\delta 0} = J_i,$$

as other terms are either 0 or not of interest for another reason. We move all time derivatives to the far right, and have the identity

$$(1+r^*)^{\delta+|I|}\partial^{I-1}\partial_t F_{i0} = (1+r^*)^{\delta+|I|}\partial^{I-1}\left(-\left(1-\frac{M\chi}{1+r^*}\right)(g\partial g F + J_i)\right). \tag{7.56}$$

After letting derivatives fall, we note that we can safely ignore the term like $(1 - M\chi/(1+r^*))^{-1}$, as any term where it is differentiated behaves very nicely (in particular, if n derivatives fall on it, we have decay like $(1+r^*)^{-n-1}$, so we have a mere reduction of order. We get similar behavior if the derivatives fall on m^*, We now look at the metric term, $h\partial_g F$. If most derivatives fall on F, or on the $\partial\tilde{m}$ part of ∂g, we can use our L^∞ estimates on ∂g. Finally, if the most derivatives fall on ∂h, we deal with this as follows: first, we note the estimate

$$\left\|(1+r^*)^{|I_1|+1/2}\partial^{I_1+1} g\right\|_{L^2(x)} \lesssim \epsilon.$$

We can use our usual L^2, L^∞ estimate, combined with a standard Sobolev estimate on F to deal with the remaining F terms (adding two spatial derivatives). We in fact have a power of $(1+r^*)$ to spare. Dealing with J is more complicated, so we write what we have so far:

$$\sum_{|I|\le k, X\in\mathbb{L}} \int_{\Sigma_0} |(\mathcal{L}^I_X \widetilde{F})(\partial_{x^{*\alpha}}, \partial_{x^{*\beta}})|^2 \lesssim \sum_{|I|\le k} (1+r^*)^{|I|}\nabla^I_x(\widetilde{F}) + \sum_{|I|\le k-1} \left\||(1+r^*)^{\delta+|I|+1}\nabla^I_{t,x}\tilde{J}(0)\right\|_2^2. \tag{7.57}$$

The $|I| \le k-1$ estimate comes from the fact that in order for $\tilde{J}$ to appear, at least one derivative must fall on F. We now deal with each of these terms. First, for the $\tilde{J}(0)$ term, we separate into $J(0) - \overline{J}(0)$ and then take our usual estimates in $\overline{J}(0)$ in order to bound the corresponding term by $|q|^2$. We now look to the $J(0)$ term. This is also treated similarly to [16] with some extra care taken to account for the metric. We need to bound the quantity

$$\sum_{|I|\le k-1} \left\| |(1+r^*)^{\delta+|I|+1} \nabla^I_{t,x} J(0) \right\|_2^2 .$$

In order to do this, we write $J(0)$ as $\phi\overline{D\phi}$. It suffices to bound

$$\sum_{|I_1|+|I_2|\le k-1} \left\| |(1+r^*)^{\delta+|I|+1} D^{I_1}\phi \overline{D^{I_2} D\phi} \right\|_2^2 .$$

In order to take care of this, we first take our time-slice Sobolev estimate

$$\sum_{X_1\in\{\partial_{x^*\alpha}\}} |(1+r^*)^{1+|I_1|} D^{I_1}_{X_1}\phi| \lesssim \sum_{\substack{|I_2|\le 2\\ X_2\in\{\partial_{x^*i}\}, X_1\in\{\partial_{x^*\alpha}\}}} \left\| (1+r)^{\delta-1+|I_1|+|I_2|} D^{I_2}_{X_2} D^{I_1}_{X_1}\phi \right\|_2 , \tag{7.58}$$

which follows from the estimate $\delta > 1/2$. This is almost equal to our initial norm; however, we must remove all (except possibly one) time derivatives with space derivatives. Our method for doing this involves commuting all time derivatives to the right, then using $\Box^{\mathbb{C}}_g\phi = 0$, recast as

$$D_t^2\phi = -\left(1-\frac{M\chi}{1+r}\right) g^{ij} D_i D_j \phi. \tag{7.59}$$

We now look at terms containing this. We have the estimate

$$\sum_{|I|\le k-2} (1+r^*)^{\delta+1+|I|} |D^I_X D^2_t\phi| = \sum_{|I|\le k-2} (1+r^*)^{\delta+1+|I|} \left| D^I_X \left(-\left(1-\frac{M\chi}{1+r}\right) g^{ij} D_i D_j\phi\right)\right| \tag{7.60}$$

This is fortunately easy to deal with, as we can take the following L^∞ estimate on g:

$$\left|\partial^{I_1}\left(-\left(1-\frac{M\chi}{1+r}\right) g^{ij}\right)\right| \lesssim \epsilon(1+t)^{-|I|-1+\iota}.$$

This follows from the standard weighted time-slice Sobolev estimate here since at most $k-2$ derivatives fall on g. Therefore, the corresponding error terms can be ignored.

When we commute time derivatives, we use the identity

$$\left|D^{I_1}[D_{t^*}, D_{x^*i}] D^{I_2}_x\phi\right| = \left|D^{I_1} F(\partial_t, \partial_{x^*i}) D^{I_2}_x\phi\right| .$$

We use the Sobolev estimate (7.58) and our usual splitting of derivatives of F to bound the corresponding terms.

Finally, we consider the estimate

$$\left\|(1+r)^{s+\gamma+|I|}\nabla_X^I(\sqrt{|g|}J^0)(0)\right\|_{L^{6/5}(x)}^2 \lesssim \mathcal{E}_k(0)+\epsilon^2. \tag{7.61}$$

The proof for this is almost identical to the proof used in [15], where we again decompose when a derivative lands on the metric term. We can fortunately commute ∇_X^I through $\sqrt{|g|}g^{\alpha\beta}$ and use the fact that g is split at time 0 along with the estimate $\sqrt{|g|}g^{00}\approx 1$. In particular, we have the estimate

$$\left\|(1+r)^{s+\gamma+|I|}\nabla_X^I(\sqrt{|g|}J^0)(0)\right\|_{L^{6/5}(x)}^2 \lesssim \sum_{|I_1|+|I_2|\leq |I|}\left\|(1+r)^{s+\gamma+|I|}D^{I_1}\phi_0\overline{D^{I_2}\dot{\phi}_0}\right\|_{L^{6/5}(x)}^2.$$

We split this up using the $L^2=L^3$ Hölder's inequality, bounding the term with most derivatives with the L^2 norm. For the L^3 norm we use the L^∞ estimates

$$(1+r)^{1/2+\delta+|J|}|D_x^J\phi_0| \lesssim \sum_{|I|\leq 1}\left\|(1+r)^{\delta+|I|+|J|}D_x^I D_x D_x^J\phi_0\right\|_{L^2(x)}, \tag{7.62a}$$

$$(1+r)^{3/2+\delta+|J|}|D_x^J\dot{\phi}_0| \lesssim \sum_{|I|\leq 2}\left\|(1+r)^{\delta+|I|+|J|}D_x^I D_x^J\dot{\phi}_0\right\|_{L^2(x)}, \tag{7.62b}$$

and then directly integrate. We show for the case when most derivatives fall on ϕ_0; the other case follows similarly.

$$\sum_{\substack{|I_1|+|I_2|\leq |I|\\ |I_1|\geq |I_2|}}\left\|(1+r)^{\delta+|I|}\nabla_1^I\phi_0\nabla^{I_2}\dot{\phi}_0\right\|_{L^{6/5}(x)} \lesssim \sum_{\substack{|I_1|+|I_2|\leq |I|\\ |I_1|\geq |I_2|}}\left\|(1+r)^{\delta+|I_1|-1}D^{I_1}\phi_0\right\|_{L^2(x)}\left\|(1+r)^{|I_2|+1}D^{I_2}\dot{\phi}_0\right\|_{L^3(x)}.$$

For the second term, we leave a factor of $(1+r)^{-1/2-\delta}$ in the L^3 norm, and bound the rest in L^∞. Since $\delta+1/2>1$,his is therefore bounded by our initial norm. It follows that

$$\mathcal{E}_k(0)^{1/2} \lesssim \|E_{0df}\|_{H^{k_0,s_0}(\mathbb{R}^3)}+\|B_0\|_{H^{k_0,s_0}(\mathbb{R}^3)}+\|D\phi_0\|_{H^{k_0,s_0}(\mathbb{R}^3)}+\left\|\dot{\phi}_0\right\|_{H^{k_0,s_0}(\mathbb{R}^3)}$$

as long as the right hand side is sufficiently small.

8 Commutator Estimates: ϕ

We attempt to establish the bound

$$\sum_{\substack{|I|\leq k\\ X\in\mathbb{L}}} \left\| \left(\Box_g^{\mathbb{C}} \mathcal{L}_X^I \phi\right) \tau_+^s \tau_-^{1/2} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k(T) + \epsilon^2, \tag{8.1a}$$

where we recall the approximation

$$w_\iota \approx \begin{cases} \tau_-^{2\iota} & r^* \leq t, \\ \tau_-^{2\delta} & r^* \geq t. \end{cases}$$

For the sake of notational simplicity, we use the following shorthand: whenever we use the sign

$$\sum_{\leq k},$$

we take it to mean the sum over all multiindexed sets of Lorentz fields $X_i \in \mathbb{L}$ and single Lorentz fields $Y_i \in \mathbb{L}$ such that there are less than or equal to k fields overall.

Due to the definition of w_ι, we can for the most part replace the w_ι with $\tau_-^{2\iota} w$, with the caveat that we need to be more precise in certain terms coming from the charge two-form $\overline{F}$. In particular, we have that $w_\iota \approx w$ in the support of $\overline{F}$, which is necessary to resolve certain terms in the exterior.

For a scalar field ϕ satisfying $\Box_g^{\mathbb{C}} \phi = 0$, we can use the identity $\mathcal{L}_X^I \Box_g^{\mathbb{C}} \phi = 0$ to reduce this to the commutator

$$\left\| \tau_+^s \tau_-^{1/2} \left(\left[\Box_g^{\mathbb{C}}, \mathcal{L}_X^I \right] \phi \right) w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k(T) + \epsilon^2. \tag{8.1b}$$

These estimates are for the most part tedious but straightforward, in the sense that we either have estimates which follow directly from [16] or estimates coming from the perturbation of the metric. For the latter, we have generally better decay in time, and we can bound these using our L^2 and L^∞ estimates on the deformation tensor.

From a qualitative viewpoint, the interesting part of the estimates coming from the deformation tensor is that we require the nice component ${}^{(X^I)}\tilde{\pi}^\dagger{}_{L^* L^*}$ to satisfy nice L^∞ estimates for some number of Lie derivatives, in the sense that they should correspond with the estimates on the term $g_{L^* L^*}$.. The other interesting part of this estimate comes from certain cancellations which are also present in the Minkowski case.

We recall the formula (2.37), which we can write as

$$[\Box_g^{\mathbb{C}}, D_Y]\phi = -D^\alpha\phi\nabla_\alpha(\nabla\cdot Y) + D_\alpha\left({}^{(Y)}\pi^{\alpha\beta}D_\beta\phi\right) - i(\nabla^\alpha F_{Y\alpha}\phi + 2F_{Y\alpha}D^\alpha\phi). \tag{8.2}$$

Iterating and taking absolute values gives us

$$\begin{aligned}\left|[\Box_g^{\mathbb{C}}, D_X^I]\phi\right| \lesssim \sum_{\substack{|I_1|+|I_2|=|I|-1\\ X_1,X_2,Y\in\mathbb{L}}} &\left|{\mathcal{L}_{X_1}^{\mathbb{C}}}^{I_1}\left(D^\alpha D_{X_2}^{I_2}\phi\nabla_\alpha(\nabla\cdot Y)\right)\right| + \left|{\mathcal{L}_{X_1}^{\mathbb{C}}}^{I_1}\left(D_\alpha\left({}^{(Y)}\pi^{\alpha\beta}D_\beta D_{X_2}^{I_2}\phi\right)\right)\right| + \\ &+\left|{\mathcal{L}_{X_1}^{\mathbb{C}}}^{I_1}\left(i(\nabla^\alpha F_{Y\alpha}D_{X_2}^{I_2}\phi + 2F_{Y\alpha}D^\alpha D_{X_2}^{I_2}\phi)\right)\right|.\end{aligned} \tag{8.3}$$

We can replace ${}^{(Y)}\pi$ with ${}^{(Y)}\tilde{\pi}^\dagger$ at the (inconsequential) cost of replacing $|I_1|+|I_2| = |I|-1$ with $|I_1|+|I_2|\le|I|-1$, as the difference of the corresponding terms in (8.3) is

$$c_Y{\mathcal{L}_{X_1}^{\mathbb{C}}}^{I_1}\Box_g^{\mathbb{C}}D_{X_2}^{I_2}\phi.$$

This is a lower order term, with one fewer derivative, which we can bound in the same way as (8.3) by again expressing $\Box_g^{\mathbb{C}}D_{X_2}^{I_2}\phi$ as a commutator and iterating.

We look at the first term on the right of (8.3) first. We commute the Lie derivative through as follows:

$$\begin{aligned}\sum_{\le k}\left|\mathcal{L}_{X_1}^{I_1}\left(-D^\alpha D_{X_2}^{I_2}\phi\nabla_\alpha(\nabla\cdot Y)\right)\right| \lesssim &\sum_{\le k}\left|g^{\alpha\beta}D_\alpha D_{X_1}^{I_1}\phi\nabla_\beta\mathcal{L}_{X_2}^{I_2}(g^{\gamma\delta}(\widetilde{\mathcal{L}}_Y g)_{\gamma\delta})\right| + \\ &\sum_{\le k}\left|{}^{(X_1^{I_1})}{\tilde{\pi}^\dagger}^{\alpha\beta}D_\alpha D_{X_2}^{I_2}\phi\nabla_\beta\mathcal{L}_{X_3}^{I_3}(g^{\gamma\delta}(\widetilde{\mathcal{L}}_Y g)_{\gamma\delta})\right| + \\ &\sum_{\le k}\left|{}^{(X_1^{I_1})}{\pi^\dagger}^{\alpha\beta}(\mathcal{L}_{X_2}^{I_2}F)_{Y_1\alpha}D_{X_3}^{I_3}\phi\nabla_\beta\mathcal{L}_{X_4}^{I_4}(g^{\gamma\delta}(\widetilde{\mathcal{L}}_{Y_2}g)_{\gamma\delta})\right|.\end{aligned} \tag{8.4}$$

The first two terms on the right come from when the Lie derivative commutes through D. In each case, when the Lie derivative falls on g, we decompose into c_Xg and $\widetilde{\mathcal{L}}_Xg$ terms, and move the reduced Lie derivative to the second line. The third term on the right comes from the commutator $[\widetilde{\mathcal{L}}_X, D]$. Note that we did not decompose the iterated deformation tensor.

We bound this term by term. This is 0 in the Minkowski case, so for the most part we expect to have extra room (in the sense of time decay) in our estimates compared to terms which are nonzero in the Minkowski case. We look at the first term on the right hand side. The terms coming from the divergence of Y can be handled using the energy norms (7.42) and (7.43). We take the null decomposition, for which we have

$$\sum_{\leq k}\left|g^{\alpha\beta}D_\alpha D_{X_1}^{I_1}\phi\nabla_\beta\mathcal{L}_{X_2}^{I_2}(g^{\gamma\delta}(\widetilde{\mathcal{L}}_Y g)_{\gamma\delta})\right| \lesssim \sum_{\leq k}|DD_{X_1}^{I_1}\phi||\overline{\partial}\widetilde{\mathcal{L}}_{X_2}^{I_2}g|+|\overline{D}D_{X_1}^{I_1}\phi||\partial\widetilde{\mathcal{L}}_{X_2}^{I_2}g|+|H^{\underline{L}^*\underline{L}^*}||DD_{X_1}^{I_1}\phi||\partial\widetilde{\mathcal{L}}_{X_2}^{I_2}g|,$$

where $\overline{D}=D_{\mathcal{T}}$, $\overline{\partial}=\partial_{\mathcal{T}}$.

We can take our first weighted estimate. If $|I_1|\leq k-6$, we have the bound

$$\sum_{\substack{\leq k\\|I_1|\leq k-6}}\left\|\tau_+^s\tau_-^{1/2}g^{\alpha\beta}D_\alpha D_{X_1}^{I_1}\phi\nabla_\beta\mathcal{L}_{X_2}^{I_2}(g^{\gamma\delta}(\widetilde{\mathcal{L}}_Y g)_{\gamma\delta})w_\iota^{1/2}\right\|_2\lesssim \tag{8.5}$$

$$\lesssim\mathcal{E}_k^{1/2}\sum_{\leq k}\left\|(\tau_+^{s-1}\tau_-^{-s}|\overline{\partial}\mathcal{L}_{X_2}^{I_2}g|+\tau_+^{-1}|\partial\mathcal{L}_{X_2}^{I_2}g|)(w_\iota/w)^{1/2}\right\|_2,$$

which is in our energy norm for g. If $|I_2|\leq k-6$, this can be bounded by

$$\epsilon\sum_{\substack{\leq k\\|I_2|\leq k-6}}\left\|(\tau_+^{s-1+\iota}\tau_-^{-1/2}|\overline{D}D_{X_1}^{I_1}\phi|+\tau_+^{s-2+\iota}\tau_-^{1/2}|DD_{X_1}^{I_1}\phi|)w_\iota^{1/2}\right\|_2 \tag{8.6}$$

We can use $s-1+\iota\leq s-1/2-\iota$, $s-2+\iota\leq -1/2-\iota$, $\tau_-^{1/2}w_\iota^{1/2}\lesssim\tau_-^s w^{1/2}$ to bound this in our energy norm.

Now we look at the second term. We can note that we only care about terms when we require the energy norm for ${}^{(X^{I_1})}\tilde{\pi}^\dagger$ (and can use L^∞ estimates for other terms), as all other terms for which we can use our L^∞ estimates for${}^{(X^{I_1})}\tilde{\pi}^\dagger$ can be straightforwardly treated in the same way as the previous term, as ${}^{(X^{I_1})}\tilde{\pi}^\dagger$ satisfies better L^∞ estimates than g. We can take our worst L^∞ estimates on the $D\phi$ and $\nabla\cdot Y$ terms, which leaves us with the norm

$$\sum_{\leq k}\mathcal{E}_k^{1/2}\epsilon\left\|\tau_+^{s-2+\iota}\tau_-^{1/2-1-1/2-s}|{}^{(X^{I_1})}\tilde{\pi}^\dagger|w_\iota^{1/2}\right\|_2, \tag{8.7}$$

which is bounded by our energy with no problem. We note that we have more room with this energy than other terms appearing so far in our estimate because two error terms in g appear.

We now look at the terms where F appears. These are similar to deal with, for the most part, and we note that in general

$$F_{Y_1\alpha}\phi$$

satisfies the same L^∞ bounds or better than $D_\alpha\phi$, provided sufficiently few derivatives fall on these terms. This follows from our null decomposition directly.

When most derivatives fall on ${}^{(X_1^{I_1})}\tilde{\pi}^{\dagger\alpha\beta}$, we split it as usual into its constant and error parts. The constant part we deal with in the same way as the first line on the right of (8.4), and the error terms we deal with in the same way as the second line.

Next, when most derivatives fall on $\nabla_\beta \mathcal{L}_{X_4}^{I_4}(g^{\gamma\delta}(\widetilde{\mathcal{L}}_{Y_2} g)_{\gamma\delta})$, we note that the remaining terms satisfy the same bounds as the corresponding terms in first line on the right of (8.4). When most derivatives fall on

For the remaining terms, we decompose F into $\widetilde{F}+\overline{F}$ as usual. When most derivatives fall on $D_{X_3}^{I_3}\phi$ or $\overline{F}$, we use our energy norm for $D_{X_3}^{I_3}\phi$ and L^∞ norms for all other terms.. In each case, we have the inequalities

$$
\begin{aligned}
|(\mathcal{L}_X^I F)_{YL^*}| &\lesssim \mathcal{E}_k^{1/2} \tau_+^{-1/2-s} \tau_-^{s-1/2}, \\
|(\mathcal{L}_X^I F)_{YS_j^*}| &\lesssim \mathcal{E}_k^{1/2} \tau_+^{-s} \tau_-^{-1+s}, \\
|(\mathcal{L}_X^I F)_{Y\underline{L}^*}| &\lesssim \mathcal{E}_k^{1/2} \tau_-^{-1},
\end{aligned}
$$

all of which follow from our L^∞ estimates on $\overline{F}$ and $\widetilde{F}$.

Combining this with our L^∞ bounds on g give us the inequality

$$
\sum_{\substack{|I_1|+|I_2|+|I_3|+|I_4| \lesssim |I|-2 \\ |I_1|,|I_2|,|I_4| \leq k-6 \\ X_i, Y_i \in \mathbb{L}}} \left\| \tau_+^s \tau_-^{1/2} {}^{(X_1^{I_1})}\pi^{\alpha\beta} \mathcal{L}_{X_2}^{I_2} F_{Y_2\alpha} D_{X_3}^{I_3}\phi \nabla_\beta \mathcal{L}_{X_4}^{I_4}(g^{\gamma\delta}(\widetilde{\mathcal{L}}_{Y_1} g)_{\gamma\delta}) w_\iota^{1/2} \right\|_2 \lesssim \tag{8.8a}
$$

$$
\lesssim \mathcal{E}_k^{1/2} \epsilon \left\| \tau_+^{s-2+\iota} \tau_-^{-1/2} |D_{X_3}^{I_3}\phi| w_\iota^{1/2} \right\|_2
$$

This is therefore contained in our energy. Finally, we look at the case where most derivatives fall on $\widetilde{F}$. Taking the null decomposition, we see that it suffices to bound the quantity

$$
\left\| \tau_+^s \tau_-^{1/2} \tau_+^{-3+\iota} |(\mathcal{L}_X \widetilde{F})_{Y_2\underline{L}^*}| w_\iota^{1/2} \right\|_2 + \left\| \tau_+^s \tau_-^{1/2} \tau_+^{-2+\iota} \tau_-^{-1} (|(\mathcal{L}_X \widetilde{F})_{Y_2 L^*}| + |(\mathcal{L}_X \widetilde{F})_{Y_2 S^*}|) w_\iota^{1/2} \right\|_2, \tag{8.9}
$$

noting that any components we would expect to decay worse in $\partial(\nabla \cdot Y)$, F are counteracted by $H_{\mathcal{LT}}$ components which introduce a factor decaying better than τ_0. These are all bounded by our energy, so we can state our first subresult:

$$
\sum_{\leq k} \left\| \tau_+^s \tau_-^{1/2} \mathcal{L}_{X_1}^{I_1} \left(-D^\alpha D_{X_2}^{I_2}\phi \nabla_\alpha (\nabla \cdot Y) \right) w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2} \epsilon \tag{8.10}
$$

The Second Term. We now turn our attention to the second term of (8.3). We recall the reduction of

${}^{(Y)}\pi^\dagger$ to its reduced form, so in particular we need to bound

$$\sum_{\leq k} \left\| \tau_+^s \tau_-^{1/2} \mathcal{L}_{X_1}^{\mathbb{C}}{}^{I_1} \left(D_\alpha \left({}^{(Y)}\tilde{\pi}^{\dagger\alpha\beta} D_\beta D_{X_2}^{I_2} \phi \right) \right) w_\iota^{1/2} \right\|_2. \tag{8.11}$$

This is again 0 in the Minkowski metric.

We recall the commutator identity

$$[\mathcal{L}_Y^{\mathbb{C}}, D_\alpha] T^\alpha = i F_{Y\alpha} T^\alpha - \nabla_\beta (\nabla \cdot Y) T^\beta \tag{8.12}$$

We commute the complex covariant derivative through the Lie derivatives using the formula

$$\left| [\mathcal{L}_{X_1}^{\mathbb{C}}{}^{I_1}, D_\alpha] \left({}^{(Y)}\tilde{\pi}^{\dagger\alpha\beta} D_\beta D_{X_2}^{I_2} \phi \right) \right| \lesssim \sum_{|I_3|+1+|I_4|=|I_1|} \left| \mathcal{L}_{X^{I_3}}^{\mathbb{C}} [\mathcal{L}_{Y_1}^{\mathbb{C}}, D_\alpha] \mathcal{L}_{X^{I_4}}^{\mathbb{C}} \left({}^{(X_2)}\tilde{\pi}^{\dagger\alpha\beta} D_\beta D_Z^{I_2} \phi \right) \right|.$$

We first look at the term where all derivatives commute through. We wish to bound the quantity

$$\sum_{\leq k} \left\| \tau_+^s \tau_-^{1/2} D_\alpha \left({}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\alpha\beta} \mathcal{L}_{X^{I_2}}^{\mathbb{C}} (D D_{X_3}^{I_3} \phi)_\beta \right) w_\iota^{1/2} \right\|_2.$$

in the associated L^2 norm.

We can as usual commute $\mathcal{L}_{Z^{I_4}}^{\mathbb{C}}$ through to get two types of terms: first, when all derivatives commute through, we have

$$R_1 = \sum_{\leq k} \left\| \tau_+^s \tau_-^{1/2} D_\alpha \left({}^{(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\alpha\beta} D_\beta D_X^{I_4} D_X^{I_2} \phi \right) w_\iota^{1/2} \right\|_2 \tag{8.13}$$

and second, we have the commutators

$$R_2 = \sum_{\leq k} \left\| \tau_+^s \tau_-^{1/2} D_\alpha \left({}^{(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\alpha\beta} (\mathcal{L}_{X_4}^{I_4} F)_{Y_1\beta} D_{X_2}^{I_2} \phi \right) w_\iota^{1/2} \right\|_2. \tag{8.14}$$

In each case, we first take the null decomposition in β in the exterior region $r > t/2 + 1/2$ (the interior estimates are easier to show as usual since we do not need to distinguish between derivatives). Now we can look at the terms. First, we consider the case when the derivative falls on $\pi^{\alpha X}$.

First, we note that $\nabla_\alpha \pi^{\alpha \underline{L}^*}$ can be treated as a sum of nice derivatives of bad components, and bad derivatives of nice components, and other nice terms. We in particular have the following decomposition:

$$|\nabla_\alpha {}^{(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\alpha\underline{L}^*}| \lesssim \tau_-^{-1} |{}^{(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\underline{L}^*\underline{L}^*}| + \tau_+^{-1} |{}^{(X^{I_3}+Y_2)}\tilde{\pi}^\dagger| + |\overline{\partial}{}^{(X^{I_3}+Y_2)}\tilde{\pi}^\dagger| + |\partial({}^{(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\underline{L}^*\underline{L}^*})|. \tag{8.15}$$

This comes from decomposing α in the null frame and using the estimates $|\nabla_\alpha \underline{L}^{*\alpha}| \lesssim \tau_-^{-1}$ (and in fact the better estimate $|\nabla_\alpha \underline{L}^{*\alpha}| \lesssim \tau_+^{-1+\iota}\tau_-^{-\iota}$), $|\nabla_\alpha L^{*\alpha}| + |\nabla_\alpha S_j^{*\alpha}| \lesssim \tau_+^{-1}$. In general we have the estimate

$$|\nabla_\alpha {}^{(X^{I_3+Y_2})}\tilde{\pi}^{\dagger\,\alpha\mathcal{T}}| \lesssim \tau_-^{-1}|{}^{(X^{I_3+Y_2})}\tilde{\pi}^{\dagger}| + |\partial({}^{(X^{I_3+Y_2})}\tilde{\pi}^{\dagger})|, \tag{8.16}$$

so we can use the same spacetime estimates for both quantities on the right.

We can bound this quantity in spacetime using the weights on the first term on the left hand side of (2.15b). We first consider the case when most derivatives fall on π. Here, we can treat R_1 and R_2 very similarly. For these, we have

$$\sum_{\substack{\leq k \\ |I_2|\leq k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\mathcal{T}} D_{\mathcal{T}} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2 \lesssim \sum_{\leq k} \left\| {\tau_+}^{-1} {\tau_-}^{\iota} \nabla_\alpha {}^{(X^{I_1})}\tilde{\pi}^{\dagger\,\alpha\mathcal{T}} \right\|_2, \tag{8.17a}$$

$$\sum_{\substack{\leq k \\ |I_2|\leq k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\mathcal{T}} F_{Z_1\mathcal{T}} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2 \lesssim \sum_{\leq k} \left\| {\tau_+}^{-1} {\tau_-}^{\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\pi^{\alpha\mathcal{T}} \right\|_2, \tag{8.17b}$$

$$\sum_{\substack{\leq k \\ |I_2|\leq k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\underline{L}^*} D_{\underline{L}^*} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2 \lesssim \sum_{\leq k} \left\| {\tau_+}^{s-1} {\tau_-}^{\iota-s} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\underline{L}^*} \right\|_2, \tag{8.17c}$$

$$\sum_{\substack{\leq k \\ |I_2|\leq k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\underline{L}^*} F_{Z_1\underline{L}^*} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2 \lesssim \sum_{\leq k} \left\| {\tau_+}^{s-1} {\tau_-}^{\iota-s} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\underline{L}^*} \right\|_2. \tag{8.17d}$$

All of these are contained in the energy bound. When more derivatives fall on other terms, we must treat the R_1 and R_2 terms slightly. Fortunately, the terms like $D\phi$ are contained in bilinear estimates similar to [16]. First, for R_1, we have the estimates

$$\sum_{\substack{\leq k \\ |I_1|+1\leq k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\mathcal{T}} D_{\mathcal{T}} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2 \lesssim \sum_{\leq k} \left\| {\tau_+}^{s-1+\iota} {\tau_-}^{-1/2+\iota} D_{\mathcal{T}} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2, \tag{8.18a}$$

$$\sum_{\substack{\leq k \\ |I_1|+1\leq k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\underline{L}^*} D_{\underline{L}^*} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2 \lesssim \sum_{\leq k} \left\| {\tau_+}^{s-1-\gamma'+\iota} {\tau_-}^{1/2+\iota} D_{\underline{L}^*} D_{X_2}^{I_2}\phi w^{1/2} \right\|_2. \tag{8.18b}$$

Both of these are straightforwardly bounded by our energy. As for the other terms, we can use our L^∞ estimate on $D_Z^{I_4+I_2}\phi$, as two vector fields necessarily appear elsewhere (and consequently ignore the charge portion of F, as we can use our energy estimate for ${}^{(X^{I_1}+Y_1)}\tilde{\pi}^\dagger$):

$$\sum_{\substack{\le k \\ |I_1|+1\le k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\mathcal{T}} F_{Z_1\mathcal{T}} D_Z^{I_4+I_2}\phi w^{1/2} \right\|_2 \lesssim \epsilon \mathcal{E}_k^{1/2} \left\| \tau_+^{s-2+\iota}\tau_-^{\iota-s}(\tau_-|F|+\tau_+|\not{F}|)w^{1/2} \right\|_2, \tag{8.18c}$$

$$\sum_{\substack{\le k \\ |I_1|+1\le k-6}} \left\| {\tau_+}^s {\tau_-}^{1/2+\iota} \nabla_\alpha {}^{(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\alpha\underline{L}^*} D_{X_2}^{I_2}\phi(\mathcal{L}_{Z_4}^{I_4}F)_{Z_1\underline{L}^*} w^{1/2} \right\|_2 \lesssim \epsilon \mathcal{E}_k^{1/2} \left\| \tau_+^{s-1-\gamma'+\iota}{\tau_-}^{1+\iota-s}|F| \right\|_2. \tag{8.18d}$$

Here we have used the estimate $|\partial h| \lesssim \epsilon {\tau_+}^{-1+\iota}{\tau_-}^{-1}$, as well as $\nabla_\alpha \tilde{\pi}^{\alpha\underline{L}^*} \lesssim \epsilon {\tau_+}^{-1-\gamma'+\iota}$. This covers all terms where the derivative falls on ${}^{(X_3)}\tilde{\pi}^\dagger$.

When the derivative falls on $D\phi$ or F, we must be careful. We first look at R_1. First, we use the standard decomposition of the null frame in terms of Lorentz fields to get the following in the extended exterior:

$$|D_{L^*}D_{\mathcal{T}}D_X^I\phi| \lesssim \left|\frac{1}{\tau_+}D_{L^*}D_YD_X^I\phi\right| + \left|\frac{1}{{\tau_+}^2}D_X^I\phi\right| \tag{8.19a}$$

$$|D_{L^*}D_{\underline{L}^*}D_X^I\phi| \lesssim \left|\frac{1}{\tau_-}D_{L^*}D_YD_X^I\phi\right| \tag{8.19b}$$

$$|D_{B_j^*}D_{\mathcal{T}}D_X^I\phi| \lesssim \left|\frac{1}{\tau_+}D_{B_j^*}D_YD_X^I\phi\right| + \left|\frac{1}{{\tau_+}^2}D_YD_X^I\phi\right| \tag{8.19c}$$

$$|D_{B_j^*}D_{\underline{L}^*}D_X^I\phi| \lesssim \left|\frac{1}{\tau_-}D_{B_j^*}D_YD_X^I\phi\right| + \left|\frac{1}{\tau_-\tau_+}D_YD_X^I\phi\right| \tag{8.19d}$$

$$|D_{\underline{L}^*}D_{\mathcal{T}}D_X^I\phi| \lesssim \left|\frac{1}{\tau_+}D_{\underline{L}^*}D_YD_X^I\phi\right| + \left|\frac{1}{{\tau_+}^2}D_YD_X^I\phi\right| \tag{8.19e}$$

$$|D_{\underline{L}^*}D_{\underline{L}^*}D_X^I\phi| \lesssim \left|\frac{1}{\tau_-}D_{\underline{L}^*}D_YD_X^I\phi\right| \tag{8.19f}$$

In all cases where a nice derivative falls on $D_{\underline{L}^*}\phi$, we commute instead. In particular,

$$D_{\mathcal{T}}D_{\underline{L}^*}\psi = D_{\underline{L}^*}D_{\mathcal{T}}\psi + D_{[\mathcal{T},\underline{L}^*]}\psi + iF_{\mathcal{T}\underline{L}^*}\psi.$$

The middle (commutator) term on the right corresponds to either 0 or $\frac{1}{r^*}S_j^*$.

We first take a look at the case where most derivatives fall on $D^2\phi$, as that is easier to work with. First, we note that the total number of derivatives isn't an issue, as we take one derivative in establishing the deformation tensor, so we have $|I| \le k-1$ in (8.19). In any term where D falls on a nice derivative, we can take the uniform bound

$$\left\|\tau_+{}^s\tau_-{}^{1/2+\iota(X^{I_1})}\pi D D_{\mathcal{T}} D^{I_2}_{X_2}\phi w^{1/2}\right\| \lesssim \left\|\tau_+{}^{s-2+\iota}\tau_-{}^{1/2+\iota} D D_Y D^{I_2}_{X_2}\phi w^{1/2}\right\|_2 + \left\|\tau_+{}^{s-3+\iota}\tau_-{}^{1/2+\iota} D_Y D^{I_2}_{X_2}\phi w^{1/2}\right\|_2.$$

Both of these are contained in our energy. When D falls on $D_{\underline{L}^*}$, we take the commutator as mentioned previously for good derivatives, noting that the only term whose energy isn't immediately bounded is the term with F (note that we cannot use the pure L^∞ bound for the ϕ term). We can take our pure L^∞ bound on F, as it is undifferentiated. This lets us bound the corresponding term by

$$\mathcal{E}_k^{1/2}\epsilon\left\|\tau_+^{s-2+\iota}\tau_-^{-1/2+\iota}|D^{I_2}_{Z_2}\phi|w^{1/2}\right\|_2.$$

This is contained in our energy.

In the case with two bad derivatives, we have the nicer estimate on ${}^{(X_3)}\tilde{\pi}^\dagger$,

$$\epsilon\left\|\tau_+^{s-1-\gamma'+\iota}\tau_-^{-1/2-\iota+\gamma'} D D_Y D^I_X\phi w^{1/2}\right\| \lesssim \epsilon\mathcal{E}_k^{1/2}$$

When most derivatives fall on π, we first take the following L^∞ estimates (with $|I| \le k-7$):

$$|D_{L^*} D_{\mathcal{T}} D^I_X\phi| \lesssim \mathcal{E}_k^{1/2}\tau_+^{-2-s}\tau_-^{-1/2} \tag{8.20a}$$

$$|D_{L^*} D_{\underline{L}^*} D^I_X\phi| \lesssim \mathcal{E}_k^{1/2}\tau_+^{-1}\tau_-^{-1/2-s} \tag{8.20b}$$

$$|D_{B^*_j} D_{\mathcal{T}} D^I_X\phi| \lesssim \mathcal{E}_k^{1/2}\tau_+^{-2-s}\tau_-^{-1/2} \tag{8.20c}$$

$$|D_{B^*_j} D_{\underline{L}^*} D^I_X\phi| \lesssim \mathcal{E}_k^{1/2}\tau_+^{-2}\tau_-^{-1/2-s} \tag{8.20d}$$

$$|D_{\underline{L}^*} D_{\mathcal{T}} D^I_X\phi| \lesssim \mathcal{E}_k^{1/2}\tau_+^{-2}\tau_-^{-1/2-s} \tag{8.20e}$$

$$|D_{\underline{L}^*} D_{\underline{L}^*} D^I_X\phi| \lesssim \mathcal{E}_k^{1/2}\tau_+^{-1}\tau_-^{-3/2-s} \tag{8.20f}$$

In particular, for all terms except the worst, we have decay like $\tau_+{}^{-2}\tau_-{}^{-1/2-s}$ or better. For these, we use the standard energy estimate. In particular, straightforward calculation gives us:

$$\sum_{\le k}\left\|\tau_+{}^s\tau_-{}^{1/2+\iota(X^{I_1})}\tilde{\pi}^{\dagger\,\mathcal{U}\mathcal{T}} D^2_{\mathcal{U},\mathcal{T}} D^{I_2}_{X_2}\phi w^{1/2}\right\|_2 \lesssim \epsilon\mathcal{E}_k^{1/2}, \tag{8.21}$$

which follows from the inequalities $s-2 < -1-\iota$, $1/2+\iota-1/2-s < -1/2-\iota$. For the case where both derivatives are bad we take the inequalities $s-1<-\iota$, $-s+1/2+\iota<0$ to get

$$\sum_{\leq k}\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(Z^{I_1})}\tilde{\pi}^{\dagger\underline{L}^*\underline{L}^*}D^2_{\underline{L}^*}D^{I_2}_Z\phi w^{1/2}\right\|_2\lesssim\mathcal{E}_k^{1/2}\sum_{\leq k}\left\|{\tau_+}^{s-1}{\tau_-}^{-3/2-s+1/2+\iota(Z^{I_1})}\pi^{\underline{L}^*\underline{L}^*}\right\|_2 \lesssim\epsilon\mathcal{E}_k^{1/2}. \tag{8.22}$$

We now look at the corresponding case in R_2. Note that here we are considering only the case where D falls on $F_{Z_1\beta}D^{I_2}_Z\phi$, as we have already covered the other case. First, when the derivative falls on ϕ, we can treat it in a similar way. We need to bound

$$\sum_{\leq k}\sum\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\alpha\beta}F_{Y_1\beta}D_\alpha D^{I_2}_{X_2}\phi w^{1/2}\right\|_2. \tag{8.23}$$

First, when most derivatives fall on $\tilde{\pi}$, we again separate in our null frame. We have

$$\sum_{\substack{\leq k\\|I_2|\leq k-6}}\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\mathcal{T}\mathcal{U}}F_{Y_1\mathcal{U}}D_\mathcal{T}D^{I_2}_{X_2}\phi w^{1/2}\right\|_2\lesssim\mathcal{E}_k^{1/2}\sum_{\leq k}\left\|\tau_+^{-1}\tau_-^{-1+\iota(X^{I_3}+Y_2)}\tilde{\pi}^\dagger\right\|_2 \tag{8.24a}$$

$$\sum_{\substack{\leq k\\|I_2|\leq k-6}}\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\underline{L}^*\mathcal{T}}F_{Y_1\mathcal{T}}D_{\underline{L}^*}D^{I_2}_{X_2}\phi w^{1/2}\right\|_2\lesssim\mathcal{E}_k^{1/2}\sum_{\leq k}\left\|\tau_+^{-1}{\tau_-}^{-1+\iota(X^{I_3}+Y_2)}\tilde{\pi}^\dagger\right\|_2 \tag{8.24b}$$

$$\sum_{\substack{\leq k\\|I_2|\leq k-6}}\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\underline{L}^*\underline{L}^*}F_{Y_1\underline{L}^*}D_{\underline{L}^*}D^{I_2}_{X_2}\phi w^{1/2}\right\|_2\lesssim\mathcal{E}_k^{1/2}\sum_{\leq k}\left\|\tau_+^{s-1}\tau_-^{-1-s+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\underline{L}^*\underline{L}^*}\right\|_2 \tag{8.24c}$$

These are all bounded with our Hardy estimates in the usual way. We can as usual ignore terms where most derivatives fall on ϕ or $\overline{F}$, as these are contained in our L^∞ estimates, so we can consider the energy on other terms. We now look at the case where most derivatives fall on $\widetilde{F}$.

$$\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\mathcal{T}\mathcal{U}}(\mathcal{L}^{I_1}_{X_1}\widetilde{F})_{Y_1\mathcal{U}}D_\mathcal{T}D^{I_2}_{X_2}\phi w^{1/2}\right\|_2\lesssim\mathcal{E}_k^{1/2}\epsilon\left\|{\tau_+}^{-1+\iota}{\tau_-}^{\iota}|(\mathcal{L}^{I_1}_{X_1}\widetilde{F})|w^{1/2}\right\|_2, \tag{8.25a}$$

$$\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\underline{L}^*\mathcal{T}}(\mathcal{L}^{I_1}_{X_1}\widetilde{F})_{Y_1\mathcal{T}}D_{\underline{L}^*}D^{I_2}_Z\phi w^{1/2}\right\|_2\lesssim\mathcal{E}_k^{1/2}\epsilon\left\|\tau_+^{s-1+\iota}\tau_-^{-s+\iota}|\mathcal{L}^{I_1}_{X_1}\cancel{F}|w^{1/2}\right\|_2+ \tag{8.25b}$$

$$+\left\|\tau_+^{s-2+\iota}\tau_-^{1-s+\iota}(|(\mathcal{L}^{I_1}_{X_1}\widetilde{F})|)w^{1/2}\right\|_2,$$

$$\left\|{\tau_+}^s{\tau_-}^{1/2+\iota(X^{I_3}+Y_2)}\tilde{\pi}^{\dagger\underline{L}^*\underline{L}^*}(\mathcal{L}^{I_1}_{X_1}\widetilde{F})_{Y_1\underline{L}^*}D_{\underline{L}^*}D^{I_2}_Z\phi w^{1/2}\right\|_2\lesssim\mathcal{E}_k^{1/2}\epsilon\left\|\tau_+^{s-1-\gamma'+\iota}\tau_-^{\gamma'+\iota-s}|\mathcal{L}^{I_1}_{X_1}\widetilde{F}|w^{1/2}\right\|_2. \tag{8.25c}$$

Here we use the shorthand

$$|\mathcal{L}_{X_1}^{I_1}\widetilde{\not{F}}|^2 = |\alpha[\mathcal{L}_{X_1}^{I_1}\widetilde{\not{F}}]|^2 + |\rho[\mathcal{L}_{X_1}^{I_1}\widetilde{\not{F}}]|^2 + |\sigma[\mathcal{L}_{X_1}^{I_1}\widetilde{\not{F}}]|^2.$$

All of these quantities are contained in our energy. Now we look at the case where ∇_α falls on F. We note that we are only looking at decomposed terms, as we have taken the null decomposition in the contraction with $\tilde{\pi}^\dagger$:

$$Z(F_{XY}) = (\mathcal{L}_Z F)_{XY} + F([Z,X],Y) + F(X,[Z,Y]).$$

As usual, even if most derivatives fall on ϕ or $\overline{F}$, we can bound this in L^∞ norm. We need to establish the following bounds:

$$\sum_{\leq k}\left\|\tau_+^s\tau_-^{1/2+\iota(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\dagger\mathcal{U}\mathcal{T}}|\partial(\mathcal{L}_{X_2}^{I_2}F)_{Y_2\mathcal{T}}||D_{X_3}^{I_3}\phi|w^{1/2}\right\|_2 \lesssim \epsilon\mathcal{E}_k^{1/2} \tag{8.26a}$$

$$\sum_{\leq k}\left\|\tau_+^s\tau_-^{1/2+\iota(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\dagger\mathcal{T}\mathcal{U}}|\bar{\partial}((\mathcal{L}_{X_2}^{I_2}F)_{Y_2\mathcal{U}})||D_{X_3}^{I_3}\phi|w^{1/2}\right\|_2 \lesssim \epsilon\mathcal{E}_k^{1/2} \tag{8.26b}$$

$$\sum_{\leq k}\left\|\tau_+^s\tau_-^{1/2+\iota(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\dagger\underline{L}^*\underline{L}^*}|\partial(\mathcal{L}_{X_2}^{I_2}F)_{Y_2\mathcal{U}}||D_{X_3}^{I_3}\phi|w^{1/2}\right\|_2 \lesssim \epsilon\mathcal{E}_k^{1/2} \tag{8.26c}$$

We consider the case where most derivatives fall on $\tilde{\pi}^\dagger$. We rewrite $\partial \approx \tau_-^{-1}Y, \bar{\partial} \approx \tau_+^{-1}Y$, expand everything out, and see that we the worst case for the bad components of $\tilde{\pi}^\dagger$ appears in the first equation. For this, we have the bound

$$\sum_{\substack{\leq k\\ |I_1|+1\leq k-6}}\left\|\tau_+^s\tau_-^{1/2+\iota(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\dagger\mathcal{U}\mathcal{T}}|\partial(\mathcal{L}_{X_2}^{I_2}F)_{Y_2\mathcal{T}}||D_{X_3}^{I_3}\phi|w^{1/2}\right\|_2 \lesssim \mathcal{E}_k\sum_{\leq k}\left\|\tau_+^{-1}\tau_-^{-1+\iota(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\dagger\mathcal{U}\mathcal{T}}\right\|_2,$$

which gives us the desired norms. Likewise, for our other term, we can bound

$$\sum_{\substack{\leq k\\ |I_1|+1\leq k-6}}\left\|\tau_+^s\tau_-^{1/2+\iota(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\dagger\underline{L}^*\underline{L}^*}|\partial(\mathcal{L}_{X_2}^{I_2}F)_{Y_2\mathcal{U}}||D_{X_3}^{I_3}\phi|w^{1/2}\right\|_2 \lesssim \mathcal{E}_k\sum_{\leq k}\left\|\tau_+^{s-1}\tau_-^{-1-s+\iota(X^{I_1}+Y_1)}\tilde{\pi}^{\dagger\,\dagger\underline{L}^*\underline{L}^*}\right\|_2,$$

which likewise appears. These estimates also cover the case when most derivatives fall on ϕ or $\overline{F}$, as we can still use L^∞ norms on these terms.

When most derivatives fall on $\widetilde{F}$, we have the estimates

$$\sum_{\substack{\leq k \\ |I_1|+1\leq k-6}} \left\| \tau_+^s \tau_-^{1/2+\iota (X^{I_1}+Y_1)} \tilde{\pi}^{\dagger}{}^{\dagger \mathcal{U}\mathcal{T}} (|\partial((\mathcal{L}_{X_2}^{I_1} \widetilde{F})_{Z\mathcal{T}})| + |\partial\!\!\!/((\mathcal{L}_{X_2}^{I_1} \widetilde{F})_{Z\mathcal{U}})|) |D_{X_3}^{I_3}\phi| w^{1/2} \right\|_2 \lesssim \tag{8.27}$$

$$\lesssim \sum_{\leq k} \epsilon \mathcal{E}_k^{1/2} \left\| \tau_+^{s-2+\iota} \tau_-^{1-s+\iota}](|\mathcal{L}_X^I \widetilde{F}| + \tau_0^{-1} |\mathcal{L}_X^I \widetilde{\cancel{F}}|) \right\|_2$$

These are within our energy norms.

Finally, we look at the commutator terms $[\mathcal{L}_Z^{\mathbb{C}}, D_\alpha]$. We have four types of terms we need to bound here:

$$\left\| \tau_+^s \tau_-^{1/2+\iota} (\mathcal{L}_{X_1}^{I_1} F)_{Y_1\alpha} (\mathcal{L}_{X_2}^{I_2} F)_{Y_2\beta}{}^{(Y_3+X^{I_3})} \tilde{\pi}^{\dagger}{}^{\alpha\beta} D_{X_4}^{I_4} \phi \right\|_2 \tag{8.28a}$$

$$\left\| \tau_+^s \tau_-^{1/2+\iota} (\mathcal{L}_{X_1}^{I_1} F)_{Y_1\alpha}{}^{(Y_3+X^{I_3})} \tilde{\pi}^{\dagger}{}^{\alpha\beta} D_\beta D_{X_4}^{I_4} \phi \right\|_2 \tag{8.28b}$$

$$\left\| \tau_+^s \tau_-^{1/2+\iota} \nabla_\alpha (X^I (\nabla \cdot Y)) (\mathcal{L}_{X_2}^{I_2} F)_{Y_2\beta}{}^{(Y_3+X^{I_3})} \tilde{\pi}^{\dagger}{}^{\alpha\beta} D_{X_4}^{I_4} \phi \right\|_2 \tag{8.28c}$$

$$\left\| \tau_+^s \tau_-^{1/2+\iota} \nabla_\alpha (X^I (\nabla \cdot Y))^{(Y_3+X^{I_3})} \tilde{\pi}^{\dagger}{}^{\alpha\beta} D_\beta D_{X_4}^{I_4} \phi \right\|_2 \tag{8.28d}$$

$$\tag{8.28e}$$

The first two terms can be bounded using identical estimates to the previous sections: first, if most derivatives fall on $D\phi$, we can bound it in L^∞ and take energy estimates on other terms; if most derivatives fall on either $\widetilde{F}$ or $\tilde{\pi}^\dagger$, we can take L^∞ estimates on other terms, noting that the L^∞ norm on $|F||\phi|$ matches the corresponding component estimates on $|D\phi|$.

For the last two terms, we note that each of these has two types of error terms coming from the metric, so we do not bother decomposing.

In the third and fourth terms, as usual we take the approximation $F_{Y_2\beta} \lesssim \tau_+|F|$. Expanding everything, we see that all terms appear with at least a weight like $\tau_+^{s-2+\iota}\tau_-^{-1/2+2\iota} w^{1/2}$, which is sufficient in each energy norm.

The Final Term: We now look at our last set of terms. We mention that the primary concern here lies in the fact that we have potential terms like $(\mathcal{L}_{X_1}^{I_1} F)_{Y\underline{L}^*} D_{L^*} D_{X_2}^{I_2}\phi$. In particular, the best estimates we can hope for for the $D_{L^*}\phi$ terms are like $\tau_+^{-2}\tau_-^{1/2-s}$, which does not provide enough decay in time along the light cone.

Consequently, we must show that we can cancel these as in [16]. We first recall the commutator terms which appear, of the form:

$$\sum_{\leq k} - i\mathcal{L}^{\mathbb{C}}_{X^{I_1}} \left(\nabla_\beta(g^{\alpha\beta}(F_{Y_1\alpha}))D^{I_2}_{X_2}\phi + 2F_{Y_1\alpha}g^{\alpha\beta}D_\beta(D^{I_2}_{X_2}\phi)\right). \tag{8.29}$$

We can bound these terms using the L^∞ estimate

$$\sum_{\leq k} \left|\mathcal{L}^{\mathbb{C}}_{X_1}{}^{I_1} \left(\nabla_\beta(g^{\alpha\beta}(F_{Y_1\alpha}))D^{I_2}_{X_2}\phi + 2F_{Y_1\alpha}D_\beta(g^{\alpha\beta}D^{I_2}_{X_2}\phi)\right)\right| \lesssim \tag{8.30}$$

$$\lesssim \sum_{\leq k} \left|[\mathcal{L}^{I_1}_{X_1}, \nabla_\beta](g^{\alpha\beta}F_{Y_1\alpha})D^{I_2}_{X_2}\phi\right| + \sum_{\leq k} \left|(\mathcal{L}^{I_1}_{X_1}(i_Y F))^\beta [\mathcal{L}^{\mathbb{C}}_{X_2}{}^{I_2}, D_\beta] D^{I_3}_{X_3}\phi\right| + \tag{8.31}$$

$$+ \sum_{\leq k} \left|\nabla_\beta \left({}^{(X^{I_1})}\pi^{\dagger\,\alpha\beta}(\mathcal{L}^{I_2}_{X_2} i_Y F)_\alpha\right) D^{I_3}_{X_3}\phi + 2\,{}^{(X^{I_1})}\pi^{\dagger\,\alpha\beta}(\mathcal{L}^{I_2}_{X_2} i_Y F)_\alpha D_\beta D^{I_3}_{X_3}\phi\right|, \tag{8.32}$$

$$\lesssim A + B + C. \tag{8.33}$$

We look at the three sums on the right. The first one, A, depends on the deviation of the metric from Minkowski, and thus can be regarded as an error term. The second, B, appears in the Minkowski case; however, due to the commutator, we avoid potentially problematic terms like $D_{L^*}\phi$ in favor of two F terms, which do not pose a problem. The third, C, also appears in the Minkowski case, and poses more of a problem, in the sense that we must rely on additional cancellations.

We first recall the identity

$$[\mathcal{L}_Y, \nabla_\alpha]T^\alpha = \nabla_\beta(\nabla\cdot Y)T^\beta,$$

which follows from straightforward calculation plus symmetry of the Ricci curvature tensor.

Likewise,

$$[\mathcal{L}^{\mathbb{C}}_Y, D_\alpha]\psi = iF_{Y\alpha}\psi.$$

A appeared in our earlier analysis; in particular, we can treat it identically to the third line in (8.4), so we can ignore it.

Next we consider B. We note that we do not require any special cancellation in this case, as our null decomposition falls on F, rather than on derivatives of ϕ. Additionally, we can use our L^∞ bound on $D^{I_5}_X\phi$, as at least two Lie derivatives are removed when making the commutator.

We split up Lie derivatives of g into their constant and error parts as usual. We have already dealt with the error terms in estimates on (8.28a), and therefore we can reduce this to the constant term case. Thus, we need to bound the terms like

$$\left\|\tau_+^s\tau_-^{1/2}g^{\alpha\beta}(\mathcal{L}_{X_1}^{I_1}F)_{Y_1\alpha}(\mathcal{L}_{X_2}^{I_2}F)_{Y_2\beta}D_{X_3}^{I_3}\phi w_\iota^{1/2}\right\|_2. \tag{8.34}$$

We decompose $\mathcal{L}_{X_1}^{I_1}F = \mathcal{L}_{X_1}^{I_1}\widetilde{F} + \mathcal{L}_{X_1}^{I_1}\overline{F}$.

First, in the case when both terms in F are current terms, we must bound

$$\sum_{\leq k}\mathcal{E}_k\left\|\tau_+^{s-2}\tau_-^{1/2}D_{X_3}^{I_3}\phi w_\iota^{1/2}\right\|_2. \tag{8.35}$$

Since $\overline{F}$ is supported for $r^* > t$, where we have the approxmation $w' \approx \tau_-^{-1}w$, we have the equivalence $\tau_+^{-1}\tau_-^{1/2}w_\iota \approx \tau_0 w' \lesssim \tau_0^{1/2+\iota}w'$. Therefore, this falls within our energy norm.

When one of the terms in F is not a current term, we can take our L^∞ norm on $D_{X_3}^{I_3}\phi$, as two derivatives fall elsewhere. We take the energy estimate on $\widetilde{F}$ (if both are $\widetilde{F}$, we take whichever one has more derivatives). First, if one $\widetilde{F}$ and one $\overline{F}$ term appears, we can bound this straightforwardly by

$$\mathcal{E}_k\left\|\tau_+^{s-1}\tau_-^{1-s}(|\widetilde{\not{F}}| + \tau_0|\widetilde{F}|)w_\iota^{1/2}\right\|_2. \tag{8.36}$$

. This can be easily bounded by our energy. We now look at the case with two $|\widetilde{F}|$ terms, which is the most problematic. We first, as usual, take our L^∞ norms on the ϕ term. Decomposing this gives us terms which can easily be bounded in the usual method of taking our pure L^∞ and L^2 bounds, with the exception of

$$\mathcal{E}_k^{1/2}\left\|\tau_+^{s+1}\tau_-^{1-s}|\alpha[\mathcal{L}_{X_1}^{I_1}\widetilde{F}]||\underline{\alpha}[\mathcal{L}_{X_2}^{I_2}\widetilde{F}]|w_\iota^{1/2}\right\|_2. \tag{8.37}$$

In particular, when most derivatives fall on $\underline{\alpha}$ we do not have enough decay from the L^∞ estimate. Instead, here we use the mixed $L^2(t)L^\infty(x)$ norm for α. We have

$$\mathcal{E}_k^{1/2}\left\|\tau_+^{s+1}\tau_-^{1-s}|\alpha||\underline{\alpha}|w_\iota^{1/2}\right\|_2 \lesssim \left\|\tau_+^{s+1}\tau_-^{1/2}|\alpha[\widetilde{F}]|w'^{1/2}\right\|_{L^2(t)L^\infty(x)}\left\|\tau_-^{1/2-s}|\underline{\alpha}[\widetilde{F}]|(w_\iota/w')^{1/2}\right\|_{L^\infty(t)L^2(x)}. \tag{8.38}$$

Using the relation

$$(w_\iota/w')^{1/2} \lesssim \tau_-^{1/2+2\iota},$$

and the inequality $1/2 - s + 1/2 + 2\iota \leq s$ gives us our desired bound.

Finally, we consider C. For this, we require special cancellation, due to the behavior of $D_\alpha D_Z^{I_4+I_2}\phi$.

In particular, the nice component in our null decomposition behaves the same as the angular components, which provides insufficient decay. We fortunately have nice cancellation properties, pointed out in [16] (and earlier by Shu). We first write

$$D_\alpha D_Z^{I_4+I_2}\phi = \frac{1}{r^*}D_\alpha(r^* D_Z^{I_4+I_2}\phi) - \frac{\partial_\alpha(r^*)}{r^*}D_Z^{I_4+I_2}\phi. \tag{8.39}$$

Therefore, we can write

$$C = \left(\nabla_\alpha\left(\mathcal{L}_Z^{I_3}(i_{Z_1}F)\right)^\alpha - 2\frac{\partial_\alpha(r^*)}{r^*}\left(\mathcal{L}_Z^{I_3}(i_{Z_1}F)\right)^\alpha\right)D_Z^{I_4+I_2}\phi + 2\left(\mathcal{L}_Z^{I_3}(i_{Z_1}F)\right)^\alpha\left(\frac{1}{r^*}D_\alpha\left(r^* D_Z^{I_4+I_2}\phi\right)\right). \tag{8.40}$$

We first consider the second term on the right. In the case where we have error terms coming from g in our iterated Lie derivative, the necessary estimates follow immediately from previous calculations (for instance, (8.28b)). In the case where we have a scalar multiple of g, we again note that the only term which cannot be broken up into pure L^∞ and L^2 estimates are the terms with one nice and one bad derivative, where we can again in each case use our $L^2(t)L^\infty(x)$ norm on the nice derivative and the $L^\infty(t)L^2(x)$ norm on the bad one.

We now look at the first term in C. It is first worth decomposing all terms where the Lie derivative falls on the raised metric g in our usual way. Thus, we have error terms like

$$\nabla_\alpha\left({}^{(X_1)}\tilde{\pi}^{\dagger\,\alpha Z}(\mathcal{L}_{X_2}^{I_2}F)_{Y_1 Z}\right)D_{X_3}^{I_3}\phi + {}^{(X_1)}\tilde{\pi}^{\dagger\,\alpha\beta}|\mathcal{L}_{X_2}^{I_2}F||D_{X_3}^{I_3}\phi|.$$

The first term can be bounded in our current norm similarly to (8.14). The second term is also straightforward, noting that in each term we have a power like $\tau_+^{s-2+\iota}$, with some additional decay coming from the F term when we need the energy for $\tilde{\pi}^\dagger$.

We can therefore pass the Lie derivative through the metric, so it suffices to bound

$$\sum_{\leq k}\left\|\tau_+^s\tau_-^{1/2}g^{\alpha\beta}\left(\nabla_\alpha(\mathcal{L}_{X_1}^{I_1}F)_{Y_1\beta}) - \frac{2\partial_\alpha(r^*)}{r^*}(\mathcal{L}_{X_1}^{I_1}F)_{Y_1\beta})\right)D_{X_2}^{I_2}\phi w_\iota^{1/2}\right\|_2 \tag{8.41}$$

We can divide the first term up into

$$g^{\alpha\beta}\nabla_\alpha(\mathcal{L}_{X_1}^{I_1}F)_{Y_1\beta} = J[\mathcal{L}_{X_1}^{I_1}F]_{Y_1}D_{X_2}^{I_2}\phi + (\nabla^\alpha Y_1^\beta)(\mathcal{L}_{X_1}^{I_1}F)_{\beta\alpha}D_{X_2}^{I_2}\phi. \tag{8.42}$$

For the current term, we note that if any Lie derivatives fall on F, we can use our L^∞ norm on ϕ, which allows us to nicely bound this with our current norm as follows:

$$\left\| \tau_+^s \tau_-^{1/2} J[\mathcal{L}_{X_1}^{I_1} F]_{Y_1} D_{X_2}^{I_2} \phi w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k^{1/2} \left\| \tau_+^{s-1} \tau_-^{1-s} J[\mathcal{L}_{X_1}^{I_1} F]_{Y_1} w_\iota^{1/2} \right\|_2 . \tag{8.43}$$

Decomposing Y_1 in terms of null vectors and using the relation $1 - s < 1/2$ allows us to bound this quantity by

$$\left\| J[\mathcal{L}_{X_1}^{I_1} F] \right\|_{L^2[w]} .$$

This we have bounded in the previous section.

If no Lie derivatives fall on F, we can use our L^∞ norm

$$|J_{Y_1}| \lesssim \mathcal{E}_k \tau_+^{-1-s} \tau_-^{-s},$$

and bound the $D_{X_2}^{I_2}$ term easily in the energy. Thus, it suffices to show our estimate for

$$\left((\nabla^\alpha Y_1^\beta) - \frac{2\nabla^\alpha (r^*)}{r^*} Y_1^\beta \right) (\mathcal{L}_{X_1}^{I_1} F)_{\beta\alpha} D_{X_2}^{I_2} \phi,$$

where Y_1 is any Lorentz vector field.

We lower the indices and consider the null decomposition. In general, for most terms it suffices to show that

$$\left((\nabla_\alpha Y_{1\beta}) - \frac{2\nabla_\alpha (r^*)}{r^*} Y_{1\beta} \right) Z_1^\alpha Z_2^\beta \lesssim 1,$$

for Z_1, Z_2 in our null frame, with the exception of the $F^{L^* S_j^*}$ components, for which we need more precise bounds. In particular, we wish to show the auxiliary estimate

$$\left(\nabla_\alpha Y_{1\beta} - \nabla_\beta Y_{1\alpha} - \frac{2\partial_\alpha (r^*)}{r^*} Y_{1\beta} \right) L^{*\alpha} S_i^{*\beta} \lesssim \tau_0^{1-\iota}. \tag{8.44}$$

The terms containing Christoffel symbols on the left satisfy our estimate, so we can replace ∇_α with ∂_α. Likewise, in the first term, we can pass S_1^β through the derivative. When we pass L^* through the derivative, we get an additional factor of $r^{*-1} S_i^{*\alpha}$. We can now write this as

$$L^*(Y_{1S_i^*}) - S_i^*(Y_{1L^*}) + \frac{1}{r^*} Y_{1S_i^*} - \frac{2}{r^*} Y_{1S_i^*} \lesssim \tau_0^{1-\iota} \tag{8.45}$$

We prove this for the Lorentz boost field, as other fields are easier. First, we expand as

$$L^*(g_{\Omega^*_{0j}S^*_i}) - S^*_i(g_{\Omega^*_{0j}L^*}) - \frac{1}{r^*}g_{\Omega^*_{0j}S^*_i}. \tag{8.46}$$

We can again ignore error terms coming from the metric as we are dealing with only nice derivatives. Expanding everything out gives us purely these error terms, thus, our bound. We note that we have similar estimates with the Christoffel symbols corresponding to $|\sigma|, |\rho|$ components, and slightly worse ones corresponding to $|\alpha|$. In particular, we have slight growth along the light cone of order $\tau_+^\iota \tau_-^{-1}$ coming from there. Therefore, we have

$$\left(\nabla_\alpha Y_{1\beta} - \nabla_\beta Y_{1\alpha} - \frac{2\partial_\alpha(r^*)}{r^*}Y_{1\beta}\right) F^{\alpha\beta} \lesssim \tau_+^{-1+\iota}|\underline{\alpha}| + |\rho| + |\sigma| + \tau_+^\iota \tau_-^{-\iota}|\alpha|. \tag{8.47}$$

Our estimate follows.

We can combine everything:

Theorem 8.1. *If ϕ solves $\Box_g^{\mathbb{C}}\phi = 0$, and the estimate*

$$\mathcal{E}_k[\phi, F] \leq 1,$$

then we have the estimate

$$\sum_{\leq k} \left\| \left(\Box_g^{\mathbb{C}} \mathcal{L}_X^I \phi\right) \tau_+^s \tau_-^{1/2} w_\iota^{1/2} \right\|_2 \lesssim \mathcal{E}_k + \epsilon^2. \tag{8.48}$$

9 The Bootstrap Estimate

We can now put everything together. We start out with our energy estimate for up to 11 derivatives:

We have the following iterated energy estimates, following from (3.14) and (4.1): When $\mathcal{E}_k(T) + \epsilon^2 \leq 1$, we have the estimates

$$\mathcal{E}_k[\tilde{F}](T) \lesssim \mathcal{E}_k[\tilde{F}](0) + |q|^2 + \sum_{\substack{|I|\leq 11 \\ X\in\mathbb{L}}} \left\| J[\mathcal{L}_X^I \tilde{F}] \right\|_{L^2[w]}^2, \tag{9.1}$$

$$\mathcal{E}_k[\phi](T) \lesssim \mathcal{E}_k[\phi](0) + \|F\|_{L^\infty[w]} \mathcal{E}_k[\phi](T) + \sum_{\substack{|I|\leq 11 \\ X\in\mathbb{L}}} \left\| \tau_+^s \tau_-^{1/2} \Box_g^{\mathbb{C}}(D_X^I \phi) w_\iota^{1/2} \right\|_{L^2[w]}^2. \tag{9.2}$$

It suffices to bound the right hand side by the initial condition energy norms plus something that can be moved over to the left.

By (8.48), we have that for sufficiently small $\mathcal{E}_k + \epsilon^2$, where the maximum value depends on the constant implicit in $\lesssim$, we can move

$$\sum_{\substack{|I|\leq 11\\ X\in\mathbb{L}}} \left\| \tau_+^s \tau_-^{1/2} \Box_g^{\mathbb{C}} (D_X^I \phi) w_\iota^{1/2} \right\|_{L^2[w]}^2$$

over to the left. Similarly, due to (7.3), we can move

$$\sum_{\substack{|I|\leq 11\\ X\in\mathbb{L}}} \left\| J[\mathcal{L}_X^I \tilde{F}] \right\|_{L^2[w]}^2$$

to the left. We can do the same with $\|F\|_{L^\infty[w]} \mathcal{E}_k[\phi](T)$ as long as $\|F\|_{L^\infty[w]}$ is sufficiently small. We can now state our main theorem:

Theorem 9.1. *There exists an $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, and we have the estimates* (2.14) *and* (2.15)*, as well as F, ϕ satisfying the initial data norms*

$$\|E_{0df}\|_{H^{k_0,s_0}(\mathbb{R}^3)} + \|B_0\|_{H^{k_0,s_0}(\mathbb{R}^3)} + \|D\phi_0\|_{H^{k_0,s_0}(\mathbb{R}^3)} + \left\|\dot{\phi}_0\right\|_{H^{k_0,s_0}(\mathbb{R}^3)} < \epsilon,$$

then there exists a constant C depending on ϵ_0 such that the energy $\mathcal{E}_k[F,\phi](t)$ is bounded by $C\epsilon$ for all time.

Theorem 1.1 follows from this. In particular, the L^∞ estimates follow directly from our L^∞ estimates Theorem 6.6 and 5.6, as well as the auxiliary L^∞ estimate (2.47). The estimates on the Energy-Momentum tensors follow from $L^2 - L^\infty$ estimates for (1.22a) and Hölder's inequality and our energy norms for (1.22b). In each case, the worst decay occurs for $\underline{L}^*$ along the light cone.

Appendix: Inequalities

As a preliminary step, we state and prove Kato's diamagnetic inequality, which will be useful in the estimates to follow, as it will allow us to replace the $\nabla\phi$ terms on right hand side of our Poincare- and Sobolev-type inequalities with the corresponding $D\phi$ terms. Given a complex scalar field ϕ and a vector field Z, we have the inequality

$$|Z(|\phi|)| \leq |D_Z \phi|. \tag{9.3}$$

The proof of this is straightforward and we do not include it here, as it can be found in [16]. We note that in all Sobolev-type estimates to follow, we can therefore replace all cases of $Z(\phi)$ with $D_Z(\phi)$ on the right hand side. We state our first Sobolev inequality:

Lemma 9.2. *For any $q > 2$, and for any function ϕ with sufficient regularity, we have the following inequality on the sphere $\mathbb{S}^2_r$ of radius r, as long as $r > t/2$, $r > 1/2$:*

$$\sup_{\mathbb{S}^2_r} |\chi\phi| \lesssim_q \tau_+^{-2/q} \left(\sum_{|I|\leq, Z\in\mathbb{O}} \|Z(\chi\phi)\|_{L^q(S^2_r)} \right) \tag{9.4}$$

Proof. This is for the most part a straightforward consequence of Morrey's inequality applied to two charts on the unit sphere, and scaling to the sphere of radius r (and introducing a factor of $r^{-2/q}$). Note that the conditions on r and t are not necessary, due to the presence of the cutoff ϕ. A more complete proof of this, and all Sobolev estimates to follow, can be found in [16] with only minor (straightforward) adaptations necessary. □

Lemma 9.3. *For $2 \leq q < 4$, if $r > t/4$ and $t > 1$, we have*

$$\|\chi\phi\|_{L^\infty(r^*)L^q(\mathbb{S}^2_r)} \lesssim_q \tau_+^{-1+2/q}\tau_-^{-1/2} \left(\left\|\tau_-\partial_{r^*}(\chi\phi)\right\|_{L^2(x)} + \sum_{|I|\leq 1, Z\in\mathbb{O}} \|Z(\chi\phi)\|_{L^2(x)} \right). \tag{9.5}$$

Proof. This follows from the Sobolev estimate on a cylinder, rescaled to a dyadic region. We first define the dyadic decomposition $\{\mathcal{I}_i^\pm\}$ for a given time slice Σ_t as follows:

$$\mathcal{U}_i = \left\{x : r^* > t/2, 2^i \leq |u^*| + 1 \leq 2^{i+1}\right\}. \tag{9.6}$$

We subdivide these as follows:

$$\mathcal{U}_i^+ = \left\{x : r^* > t/2, 2^i \leq |u^*| + 1 \leq 2^{i+1}, u^* > 0\right\}, \qquad \mathcal{U}_i^- = \left\{x : r^* > t/2, 2^i \leq |u^*| + 1 \leq 2^{i+1}, u^* < 0\right\}. \tag{9.7}$$

Thus, $\mathcal{U}^+$ are supported in the interior, and $\mathcal{U}^-$ are supported in the exterior. Additionally, for any given time slice, $\mathcal{U}_i^+$ is empty for sufficiently large i. We can construct a partition of unity $\{\chi_{\mathcal{U}_i^\pm}\}$ such that the support of each is in the region $\{x : 2^{i-1} \leq |u^*| \leq 2^{i+2}, r^* > t/4\}$ and derivatives satisfy the bound $\partial_{r^*}(\chi_{\mathcal{U}_i^\pm}) \lesssim 2^{-i}$ for some constant independent of i.

We now define the cylindrical region

$$(\tilde{r}, \omega) \in \mathcal{A} = [1/4, 4] \times \mathbb{S}^2. \tag{9.8}$$

We take maps from our cylinder to the region $\mathcal{U}_i^{\pm}$ as follows:

$$(\tilde{r}, \omega) \to (t, t \pm 2^i r^* \omega), \tag{9.9}$$

with an appropriate cutoff when necessary. We note that we scale the radial variable by approximately τ_- and the spherical variables by τ_+. Then we take the fractional Sobolev estimates on the region $\mathcal{A}$

$$\|\chi\phi\|_{L^\infty(\mathbb{R})} \lesssim \|\chi\phi\|_{H^{1/2+2\epsilon_1}}\ , \tag{9.10a}$$

$$\|\chi\phi\|_{L^q(\mathbb{R}^2)} \lesssim \|\chi\phi\|_{H^{1-2/q+2\epsilon_2}}\ , \tag{9.10b}$$

which hold for all $\epsilon_i > 0, 2 \leq q < 4$. Since the inequality

$$(1 + |\xi_x|^2)^{1/4+\epsilon_1}(1 + |\xi_y|^2)^{1/2-/q+\epsilon_2} \lesssim (1 + |\xi_x|^2 + |\xi_y|^2)^{1/2} \tag{9.11}$$

holds in the phase space for sufficiently small ϵ_i (depending on q), taking charts gives us the inclusion inequality

$$\|\chi\phi\|_{L^\infty(r^*)L^q(\mathbb{S}^2)} \lesssim \|\chi\phi\|_{H^1(\mathcal{A})}\ . \tag{9.12}$$

We can take our change of variables, noting scaling, to get the estimate (9.5). $\square$

This covers our estimates for the extended exterior. We now look at the far interior.

Lemma 9.4. *If $t \geq 1$, $r < 3/4t$, we have the following estimates for compactly supported functions f:*

$$\|f\|_{L^\infty(\mathbb{R}^3)} \lesssim t^{-1/2} \sum_{\substack{Z \in \{S^*, \Omega^*_{0i}\} \\ |I| \leq 1}} \left\|Z^I f\right\|_{L^6(\mathbb{R}^3)} \tag{9.13a}$$

$$\|f\|_{L^6(\mathbb{R}^3)} \lesssim t^{-1} \sum_{\substack{Z \in \{S^*, \Omega^*_{0i}\} \\ |I| \leq 1}} \left\|Z^I f\right\|_{L^2(\mathbb{R}^3)} \tag{9.13b}$$

Proof. This follows from a rescaling to the unit ball, noting that

$$t\left\|\nabla f\right\|_{L^p} \lesssim \sum_{\substack{Z \in \{S^*, \Omega^*_{0i}\} \\ |I| \leq 1}} \left\|Z^I(f)\right\|_{L^p}.$$

This follows almost identically from the proof in [16], noting that $\partial_{t^*}, \partial_{x^{*i}}$ and ∂_t, ∂_i are equivalent. $\square$

Finally, we consider the light cone estimate. As in [16], this is not strictly necessary in closing our estimate, as we can get our full results using an $L^2(t)L^\infty(x)$ estimate following from the time slice Sobolev estimates. However, this estimate gives us more precise control over the asymptotic behavior:

Lemma 9.5. *For $2 \le q < 4$, we have the global estimate*

$$\|\chi\phi\|_{L^\infty(u^*)L^q(\mathbb{S}^2_{\tilde{r}})} \lesssim \tau_+^{-3/2-2/q} \sum_{\substack{Z\in\{\underline{u}^*L^*,\mathbb{O}\}\\ |I|\le 1}} \left\|Z^I(\chi\phi)\right\|_{L^2(C(u^*))}. \tag{9.14}$$

Proof. This is similar to the proof of inequality (9.5), with two differences. First, due to boundary considerations along the light cone, we need to take a Sobolev extension function across the endpoints of the time slab $t \in [1, T]$. Second, we take our dyadic decomposition in $\underline{u}^*$ instead of u^*. This introduces a factor of τ_+ instead of τ_- in the analogue to the radial derivative $\partial_{\tilde{r}}$ in the cylinder. However, this is paired with L^*, a nicer behaving directional derivative. □

We can now put everything together:

Theorem 9.6. *Given a smooth test function ϕ, we have the following estimates:*

$$\left\|\tau_+^{1+\delta_+}\tau_-^{1/2+\delta_-}\chi\phi w^{1/2}\right\|_{L^\infty(\mathbb{R}^3)} \lesssim \sum_{\substack{|I|,|J|\le 1\\ X\in\{\tau_-\partial_{r^*}\}\cup\mathbb{O}, Y\in\mathbb{O}}} \left\|\tau_+^{\delta_+}\tau_-^{\delta_-}X^IY^J(\chi\phi)w^{1/2}\right\|_{L^2(\mathbb{R}^3)} \tag{9.15}$$

$$\left\|\tau_+^{3/2+\delta_+}\chi\phi w^{1/2}\right\|_{L^\infty(C_{u^*})} \lesssim \sum_{\substack{|I|,|J|\le 1\\ X\in\{\underline{u}^*L^*,\mathbb{O}\}, Y\in\mathbb{O}}} \left\|\tau_+^{\delta_+}X^IY^J(\chi\phi)w^{1/2}\right\|_{L^2(C_{u^*})} \tag{9.16}$$

$$\left\|\tau_+^{3/2+\delta_+}(1-\chi)\phi w^{1/2}\right\|_{L^\infty(\mathbb{R}^3)} \lesssim \sum_{\substack{|J|\le 2\\ Z\in\{S^*,\Omega^*_{0i}\}}} \left\|\tau_+^{\delta_+}Z^I((1-\chi)\phi)w^{1/2}\right\|_{L^2(\mathbb{R}^3)} \tag{9.17}$$

as well as their complex covariant equivalents

$$\left\|\tau_+^{1+\delta_+}\tau_-^{1/2+\delta_-}\chi\phi\right\|_{L^\infty(\mathbb{R}^3)} \lesssim \sum_{\substack{|I|,|J|\le 1\\ X\in\{\tau_-\partial_{r^*}\}\cup\mathbb{O}, Y\in\mathbb{O}}} \left\|\tau_+^{\delta_+}\tau_-^{\delta_-}D_X^ID_Y^J(\chi\phi)\right\|_{L^2(\mathbb{R}^3)}, \tag{9.18}$$

$$\left\|\tau_+^{3/2+\delta_+}\chi\phi\right\|_{L^\infty(C_{u^*})} \lesssim \sum_{\substack{|I|,|J|\le 1\\ X\in\{\underline{u}^*L^*,\mathbb{O}\}, Y\in\mathbb{O}}} \left\|\tau_+^{\delta_+}D_X^ID_Y^J(\chi\phi)\right\|_{L^2(C_{u^*})}, \tag{9.19}$$

$$\left\|\tau_+^{3/2+\delta_+}(1-\chi)\phi\right\|_{L^\infty(\mathbb{R}^3)} \lesssim \sum_{\substack{|J|\le 2\\ Z\in\{S^*,\Omega^*_{0i}\}}} \left\|\tau_+^{\delta_+}D_Z^I((1-\chi)\phi)\right\|_{L^2(\mathbb{R}^3)}. \tag{9.20}$$

Proof. This straightforwardly follows from (9.4)-(9.14), with powers of w and $\delta_\pm$ added during the dyadic decomposition. $\square$

We first look at a model inequality in 1+3 dimensions. The proof of this is adapted from an intermediate result found in [8], and can be readily generalized to results which will be useful in our L^2 and L^∞ estimates. We go through it in detail,

Lemma 9.7. *For any function ϕ with suitable regularity and decay at ∞, we have the inequality*

$$\int_{\Sigma_t} (t+r)^2 \left|\frac{\phi}{r}\right|^2 dx \lesssim \int_{\Sigma_t} (r-t)^2 \left|\frac{\partial_r(r\phi)}{r}\right|^2 dx. \tag{9.21}$$

We can ignore the decay at ∞ by proving it for compact functions and using its closure in relevant function spaces.

Proof. By transforming into spherical coordinates and noting that the integrating factor scales in r like r^2, we can reduce this problem to showing the inequality

$$\int_0^\infty \left(\frac{t+r}{r}\right)^2 (r\phi)^2\, dr \lesssim \int_0^\infty (r-t)^2 \partial_r(r\phi)^2\, dr, \tag{9.22}$$

where we have restricted ϕ along lines of constant ω. To show that this is true, we first take the one-dimensional inequality

$$\int_0^\infty (Cf\partial_r\psi + g\psi)^2 - \partial_r(Cfg\psi^2) \geq 0, \tag{9.23}$$

which holds as long as $fg\psi^2$ is absolutely continuous and vanishes at 0 and at ∞ This is satisfied for $\psi = r\phi$, where ϕ is compactly supported. We can think of f and g as weight functions, and C is an arbitrary constant. We can rewrite this as

$$\int_0^\infty \left(C\partial_r(fg) - g^2\right)(r\phi)^2 \lesssim \int_0^\infty C^2 f^2 (\partial_r(r\phi))^2\, dr. \tag{9.24}$$

We select

$$f = r - t,$$
$$g = \frac{r+t}{r}.$$

Then,

$$\partial_r(fg) = \frac{r^2+t^2}{r^2}.$$

Selecting $C = 4$ we see that

$$\left(C\partial_r(fg) - g^2\right) \geq \left(\frac{r+t}{r}\right)^2.$$

The inequality (9.22) follows. $\square$

Lemma 9.8. *For $\frac{1}{2} < s \leq 1$, and for compactly supported f, we have the estimate*

$$\int_{\Sigma_t} \tau_+^{2s} \left|\frac{\phi}{r}\right|^2 w\,dx \lesssim \int_{\Sigma_t} \tau_-^{2s} \left|\frac{D_{r^*}(r^*\phi)}{r}\right|^2 w\,dx. \tag{9.25}$$

Proof. First note that we can replace D_{r^*} with ∂_{r^*} without issue, due to Kato's inequality. As in the previous lemma, we reduce to the one-dimensional inequality

$$\int_0^\infty \tau_+^{2s}|\phi|^2 w\,dr \lesssim \int_0^\infty \tau_-^{2s}|\partial_{r^*}(r^*\phi)|^2 w\,dr. \tag{9.26}$$

Since dr and dr^* are equivalent, we can replace the former with the latter without issue. We now take inequality (9.24), with r^* in place of r, and

$$f = |r^* - t|^s \mathrm{sgn}(r^* - t) w^{1/2}, \tag{9.27a}$$

$$g = \frac{|r^* + t|^s}{r^*} w^{1/2}. \tag{9.27b}$$

We have that in this case is equal to

$$\partial_r^*(fg) = \mathrm{sgn}(r^* - t)\frac{2r^{*2}s|r^{*2} - t^2|^{s-1}\mathrm{sgn}(r^* - t) - |r^{*2} - t^2|^s}{r^{*2}} w + \frac{\partial_{r^*}(w)}{w}(fg).$$

The last term is strictly positive, as $\partial_{r^*}(w)$ is supported when $r^* - t > 0$. We can rewrite

$$\partial_r^*(fg) \geq \frac{((2s-1)r^{*2} + t^2)|r^{*2} - t^2|^{s-1}}{r^{*2}} w.$$

Choose C such that $(2s-1)C \geq 4$. Then, noting $s - 1 \leq 0$, it follows that

$$C\partial_r^*(fg) \geq \frac{4|r^{*2} + t^2|^s}{r^{*2}} w.$$

For $s \leq 1$, we have

$$C\partial_r^*(fg) - g^2 \geq g^2.$$

This gives us the preliminary estimate

$$\int_{\Sigma_t} |r^* + t|^{2s} \left|\frac{\phi}{r}\right|^2 w\,dx \lesssim \int_{\Sigma_t} |r^* - t|^{2s} \left|\frac{D_{r^*}(r^*\phi)}{r}\right| w\,dx. \tag{9.28}$$

We can add a time-shifted estimate replacing t with $t+1$ to get the full estimate. □

Similar reasoning gives us the inequality

$$\int_{\Sigma_t} \tau_+^{2s}\tau_0^{1+2\iota} \left|\frac{\phi}{r^*}\right|^2 (w')\,dx \lesssim \tau_-^{2s}\tau_0^{1+2\iota} \left|\frac{D_{r^*}(r^*\phi)}{r^*}\right|^2 (w')\,dx, \tag{9.29}$$

as long as we have the inequality $s + \delta > 1$.

Now we look at an estimate restated from [15] which is not strictly necessary under our assumptions, but will elucidate the energy bounds we use.

Lemma 9.9. *Let $\gamma > 1/2$ be a constant, and take the weight*

$$w_g = \begin{cases} 1 & r^* < t \\ (1 + (r^* - t))^{1+2\gamma} & r^* \geq t \end{cases}$$

Then, when $1/2 + 2\iota < \gamma$, $0 \leq \kappa \leq 1$, the following inequalities hold for ϕ with sufficient decay at ∞:

$$\left\|\tau_-^{-3/2-\iota}\tau_+^{-\iota}\phi w_g^{1/2}\right\|_2 \lesssim \left\|\tau_-^{-1/2-\iota}\tau_+^{-\iota}\partial_{r^*}\phi w_g^{1/2}\right\|_2 \tag{9.30a}$$

$$\left\|\tau_-^{-1}\tau_+^{-1/2-2\iota}\phi w_g^{1/2}\right\|_2 \lesssim \left\|\tau_+^{-1/2-2\iota}\partial_{r^*}\phi w_g^{1/2}\right\|_2 \tag{9.30b}$$

Proof. These both follow from Lemma 13.1 in [15]. However, we can use (9.24) to present a slightly simpler version. We prove inequality (9.30a) here and leave (9.30b) as an exercise for the reader. We set:

$$f = (1 + |r^* - t|)^{-1/2-\iota}|1 + r^* + t|^{-\iota} r^* w_g^{1/2}$$
$$g = (1 + |r^* - t|)^{-3/2-\iota}|1 + r^* + t|^{-\iota} r^* w_g^{1/2}$$

Then,

$$\partial_{r^*}(fg) = \begin{cases} \partial_{r^*}\left((1 + r^* - t)^{2\gamma-1-2\iota}(1 + r^* + t)^{-2\iota} r^{*2}\right) & r^* > t, \\ \partial_{r^*}\left((1 + t - r^*)^{-2-2\iota}(1 + r^* + t)^{-2\iota} r^{*2}\right) & r^* \leq t. \end{cases}$$

For $r^* > t$, we can write this derivative as

$$\partial_{r^*}(fg) = \left(\frac{2}{r^*} + \frac{2\gamma - 1 - 2\iota}{1 + r^* - t} - \frac{2\iota}{1 + r^* + t}\right) fg \geq 2\iota g^2, \tag{9.31}$$

where we have used $2 - \iota > 0$, $2\gamma - 1 - 2\iota > 2\iota$. Likewise, for $r^* < t$, we have

$$\partial_{r^*}(fg) = \left(\frac{2}{r^*} + \frac{2 + 2\iota}{1 + t - r^*} - \frac{2\iota}{1 + r^* + t}\right) fg \geq 2\iota g^2, \tag{9.32}$$

which again satisfies our inequality. □

We now prove an estimate along the same lines which is better suited to our conformal Morawetz estimate.

Lemma 9.10. *For p, q such that $p > -1$, $|q| < p + 1$, and for test functions ϕ, we have the inequality*

$$\int_{\Sigma_t} \tau_-^p \tau_+^q |\phi|^2 w \, dx \lesssim \int_{\Sigma_t} \tau_-^{p+2} \tau_+^q \left|\frac{D_{r^*}(r^*\phi)}{r^*}\right|^2 w \, dx. \tag{9.33}$$

Proof. First, note that this is as usual equivalent to the one-dimensional inequality

$$\int_0^\infty \tau_-^p \tau_+^q |r^*\phi|^2 w \, dr^* \lesssim \int_0^\infty \tau_-^{p+2} \tau_+^q |\partial_{r^*}(r^*\phi)|^2 w \, dr^*. \tag{9.34}$$

Additionally, we can replace τ_- and τ_+ with $1 + |r^* - t|$ and $1 + r^* + t$ respectively. We take as usual

$$f = (1 + |r^* - t|)^{p/2+1} \operatorname{sgn}(r^* - t)(1 + r^* + t)^{q/2} w^{1/2}$$
$$g = (1 + |r^* - t|)^{p/2} (1 + r^* + t)^{q/2} w^{1/2}$$

Then,

$$\partial_{r^*}(fg) = \begin{cases} \left(\frac{p+1+2\delta}{1+|r^*-t|} + \frac{q}{1+r^*+t}\right) fg & r^* > t, \\ \left(\frac{p+1}{1+|r^*-t|} - \frac{q}{1+r^*+t}\right) fg & r^* < t. \end{cases}$$

Since $g^2 = fg(1 + |r^* - t|)^{-1}$ and $1 + |r^* - t| < 1 + r^* + t$, it follows that

$$\partial_{r^*}(fg) \geq (p + 1 - |q|) g^2.$$

The result follows. □

References

[1] Lars Andersson and Pieter Blue. "Uniform energy bound and asymptotics for the Maxwell field on a slowly rotating Kerr black hole exterior". In: *J. Hyperbolic Differ. Equ.* 12.4 (2015), pp. 689–743. ISSN: 0219-8916. URL: https://doi.org/10.1142/S0219891615500204.

[2] Lydia Bieri, Shuang Miao, and Sohrab Shahshahani. "Asymptotic properties of solutions of the Maxwell Klein Gordon equation with small data". In: *Comm. Anal. Geom.* 25.1 (2017), pp. 25–96. ISSN: 1019-8385. DOI: 10.4310/CAG.2017.v25.n1.a2. URL: http://dx.doi.org/10.4310/CAG.2017.v25.n1.a2.

[3] Pieter Blue and Jacob Sterbenz. "Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space". In: *Comm. Math. Phys.* 268.2 (2006), pp. 481–504. ISSN: 0010-3616. DOI: 10.1007/s00220-006-0101-6. URL: http://dx.doi.org/10.1007/s00220-006-0101-6.

[4] D. Christodoulou and S. Klainerman. "Asymptotic properties of linear field equations in Minkowski space". In: *Comm. Pure Appl. Math.* 43.2 (1990), pp. 137–199. ISSN: 0010-3640. DOI: 10.1002/cpa.3160430202. URL: http://dx.doi.org/10.1002/cpa.3160430202.

[5] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*. Vol. 41. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993, pp. x+514. ISBN: 0-691-08777-6.

[6] Mihalis Dafermos and Igor Rodnianski. "A new physical-space approach to decay for the wave equation with applications to black hole spacetimes". In: *XVIth International Congress on Mathematical Physics*. World Sci. Publ., Hackensack, NJ, 2010, pp. 421–432. DOI: 10.1142/9789814304634_0032. URL: http://dx.doi.org/10.1142/9789814304634_0032.

[7] Douglas M. Eardley and Vincent Moncrief. "The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space." In: *Comm. Math. Phys.* 83.2 (1982), pp. 171–212. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1103920801.

[8] Lars Hörmander. *Lectures on nonlinear hyperbolic differential equations*. Vol. 26. Mathématiques & Applications (Berlin) [Mathematics & Applications]. Springer-Verlag, Berlin, 1997, pp. viii+289. ISBN: 3-540-62921-1.

[9] C. Kauffman. "Global Stability for Charged Scalar Fields in an Asymptotically Flat Metric in Harmonic Gauge". In: *ArXiv e-prints* (Jan. 2018). arXiv: 1801.09648 [math.AP].

[10] S. Klainerman and M. Machedon. “On the Maxwell-Klein-Gordon equation with finite energy”. In: *Duke Math. J.* 74.1 (1994), pp. 19–44. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-94-07402-4. URL: http://dx.doi.org/10.1215/S0012-7094-94-07402-4.

[11] H. Lindblad and V. Schlue. “Scattering from infinity for semilinear models of Einstein’s equations satisfying the weak null condition”. In: *ArXiv e-prints* (Nov. 2017). arXiv: 1711.00822 [math.AP].

[12] H. Lindblad and M. Tohaneanu. “Global existence for quasilinear wave equations close to Schwarzschild”. In: *ArXiv e-prints* (Oct. 2016). arXiv: 1610.00674 [math.AP].

[13] Hans Lindblad. “On the asymptotic behavior of solutions to the Einstein vacuum equations in wave coordinates”. In: *Comm. Math. Phys.* 353.1 (2017), pp. 135–184. ISSN: 0010-3616. DOI: 10.1007/s00220-017-2876-z. URL: http://dx.doi.org/10.1007/s00220-017-2876-z.

[14] Hans Lindblad and Igor Rodnianski. “Global existence for the Einstein vacuum equations in wave coordinates”. In: *Comm. Math. Phys.* 256.1 (2005), pp. 43–110. ISSN: 0010-3616. DOI: 10.1007/s00220-004-1281-6. URL: http://dx.doi.org/10.1007/s00220-004-1281-6.

[15] Hans Lindblad and Igor Rodnianski. “The global stability of Minkowski space-time in harmonic gauge”. In: *Ann. of Math. (2)* 171.3 (2010), pp. 1401–1477. ISSN: 0003-486X. DOI: 10.4007/annals.2010.171.1401. URL: http://dx.doi.org/10.4007/annals.2010.171.1401.

[16] Hans Lindblad and Jacob Sterbenz. “Global stability for charged-scalar fields on Minkowski space”. In: *IMRP Int. Math. Res. Pap.* (2006), Art. ID 52976, 109. ISSN: 1687-3017.

[17] Julien Loizelet. “Problèmes globaux en relativité générale”. Universitè Francois Rabelais, Tours, France. PhD thesis. 2008. URL: http://www.theses.fr/2008TOUR4006.

[18] Georgios Moschidis. “The r^p-weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications”. In: *Ann. PDE* 2.1 (2016), Art. 6, 194. ISSN: 2199-2576. URL: https://doi.org/10.1007/s40818-016-0011-7.

[19] J. Oliver and J. Sterbenz. “A Vector Field Method for Radiating Black Hole Spacetimes”. In: *ArXiv e-prints* (May 2017). arXiv: 1705.10714 [math.AP].

[20] Jesús Oliver. “A vector field method for non-trapping, radiating spacetimes”. In: *J. Hyperbolic Differ. Equ.* 13.4 (2016), pp. 735–790. ISSN: 0219-8916. URL: https://doi.org/10.1142/S021989161650020X.

[21] Volker Schlue. “Decay of linear waves on higher-dimensional Schwarzschild black holes”. In: *Anal. PDE* 6.3 (2013), pp. 515–600. ISSN: 2157-5045. URL: https://doi.org/10.2140/apde.2013.6.515.

[22] Jared Speck. "The nonlinear stability of the trivial solution to the Maxwell-Born-Infeld system". In: *J. Math. Phys.* 53.8 (2012), pp. 083703, 83. ISSN: 0022-2488. DOI: 10.1063/1.4740047. URL: http://dx.doi.org/10.1063/1.4740047.

[23] Jacob Sterbenz and Daniel Tataru. "Local energy decay for Maxwell fields Part I: Spherically symmetric black-hole backgrounds". In: *Int. Math. Res. Not. IMRN* 11 (2015), pp. 3298–3342. ISSN: 1073-7928.

[24] S. Yang. "Decay of solutions of Maxwell-Klein-Gordon equations with large Maxwell field". In: *ArXiv e-prints* (Nov. 2015). arXiv: 1511.00251 [math.AP].

[25] S. Yang. "On global behavior of solutions of the Maxwell-Klein-Gordon equations". In: *ArXiv e-prints* (Nov. 2015). arXiv: 1511.00250 [math.AP].

[26] Nina Zipser. *The global nonlinear stability of the trivial solution of the Einstein-Maxwell equations.* Thesis (Ph.D.)–Harvard University. ProQuest LLC, Ann Arbor, MI, 2000, p. 198. ISBN: 978-0599-77776-7. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:9972467.

Vitae

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